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A new algorithm for nonlinear fractional BVPs

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ABSTRACT

In this letter, an efficient numerical scheme is discussed to solve the nonlinear differential equation of fractional order with boundary conditions. The technique relies on the Quasi-Newton's method (QNM) and the simplified reproducing kernel method (SRKM). Two numerical examples are considered to confirm the performance of the proposed method and comparisons with the existing solution techniques are reported as well. Numerical results faithfully reveal the accuracy, efficiency and simplicity of the method.

1. Introduction

In recent years, numerous applications of nonlinear boundary value problems for fractional differential equations (FBVPs) have occurred in many branches of science and engineering such as bioengineering, biophysics, control theory, finance and signal processing. The fractional-order models describe natural phenomenons more practical and realistic than the classical integer-order models because of their excellent instrument for the description of memory and hereditary properties.

Motivated by the consequent growing applications of FBVPs, increasing interests have been given to the development of analytical and efficient numerical techniques concerning with such problems. For instance, He [1] was the first to propose the variational iteration method and homotopy perturbation method for finding the solutions of linear and nonlinear problems. These methods have been successfully extended by many authors [2,3] for finding the analytical approximate solutions as well as numerical approximate solutions of fractional differential equations. On the other hand, the monotone iterative technique, combined with the method of lower and upper solutions [4–7] were introduced to study the FBVPs. Momani and Aslam Noor [8] established the implementation of ADM to derive analytic approximate solutions of the linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations.

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In this letter, a novel iterative scheme called Quasi-Newton's method [9] is extended to solve the following nonlinear FBVPs:

$$\begin{cases} D^{\alpha}u(x) + p(x)\mathcal{N}(u) = f(x), & x \in [0,1], \\ u(0) = \lambda_0, \qquad u(1) = \lambda_1, & u'(0) = \lambda_2 \end{cases} \tag{1}$$

where p, f are continuous functions, $\mathcal{N}$ is nonlinear, and D^{α} is the Caputo derivative of order α. Here we consider the case $2 < \alpha < 3$.

The remainder of the present letter is arranged as follows. In Section 2, some essential definitions about fractional calculus and reproducing kernel spaces are recalled. The description of Quasi-Newton's method is given in Section 3. In Section 4, we simplify the reproducing kernel method to solve the linear equations converted by the Quasi-Newton's method. Numerical experiments are provided to verify the efficiency and reliability of the proposed technique in Section 5. Finally, we end this letter with a brief conclusion in Section 6.

2. Preliminaries

In this section, some basic concepts and preliminary definitions are introduced for our subsequent development.

Definition 2.1. The fractional order α of function $u(x)$, in Caputo's sense, is defined as

$$D^{\alpha}u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} u^{(n)}(t)dt,$$

where $\Gamma(\cdot)$ is Gamma function and $n = [\alpha] + 1$, $[\alpha]$ denotes the greatest integer smaller than α.

Definition 2.2. $W_2^m[0,1] = \{u(x) |\, u^{m-1}(x) \; is \; an \; absolutely \; continuous \; real \; value \; function \; in \; [0,1],\; u^{(m)}(x) \in L^2[0,1]\}$. The inner product is equipped by

$$\langle u(x), v(x) \rangle = \sum_{k=0}^{m-1} u^{(k)}(0)v^{(k)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x)dx.$$

From [10], it follows that $W_2^4\,[0,1]$ and $W_2^1\,[0,1]$ are reproducing kernel spaces and their kernels are denoted as $R_x(y)$ and $r_x(y)$ respectively.

3. The QNM for Eq. (3)

3.1. Fréchet derivative

Definition 3.1. Let $\mathcal{F} : X \to Y$, where X and Y are Banach spaces. Then a bounded and linear operator $\mathcal{A} : X \to Y$ is called a *Fréchet derivative* of $\mathcal{F}$ at $u_0 \in X$ if

$$\lim_{h \to 0} \frac{\| \mathcal{F}(u_0 + h) - \mathcal{F}(u_0) - \mathcal{A}(h) \|_Y}{\|h\|_X} = 0$$

for all $h \in X$, it is denoted by $\mathcal{F}'(u_0)$ commonly.

By Eq. (3), we define an operator $\mathcal{F} : W_2^4[0,1] \to W_2^1[0,1]$, satisfying

$$\mathcal{F}(u) = D^{\alpha}(u) + p(x)\mathcal{N}(u). \tag{2}$$

Therefore, Eq. (3) can be converted into an operator equation as follows

$$\begin{cases} \mathcal{F}(u) = f(x), & x \in [0,1] \\ u(0) = \lambda_0, \qquad u(1) = \lambda_1, & u'(0) = \lambda_2. \end{cases} \tag{3}$$