



On a problem of S.L. Sobolev



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ABSTRACT

In his famous works of 1930 [1,2] Sergey L. Sobolev has proposed a construction of the solution of the Cauchy problem for the hyperbolic equation of the second order with variable coefficients in \mathbb{R}_s . Although Sobolev did not construct the fundamental solution, his construction was modified later by Romanov (2002) and Smirnov (1964) to obtain the fundamental solution. However, these works impose a restrictive assumption of the regularity of geodesic lines in a large domain. In addition, it is unclear how to realize those methods numerically. In this paper a simple construction of a function, which is associated in a clear way with the fundamental solution of the acoustic equation with the variable speed in 3-d, is proposed. Conditions on geodesic lines are not imposed. An important feature of this construction is that it lends itself to effective computations.

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1. Introduction

In 1930 Sergey L. Sobolev, one of the distinguished mathematicians of the 20th century, has published two works [1,2] where he has constructed the solution of the hyperbolic equation of the second order in the 3-d case with variable coefficients in the principle part of the operator. It was assumed that these coefficients depend on spatial variables. This result is readily available in the textbook of Smirnov [3]. Sobolev did not find the fundamental solution. The main reason of this was that the notion of the fundamental solution was unknown in 1930. Still, in his books about inverse problems [4,5] Romanov has modified the method of Sobolev to construct the fundamental solution of that equation. However, the constructions of both Sobolev and Romanov impose a quite restrictive assumption on the variable coefficients in the principal part of the hyperbolic operator. Let t be time and let $t \in (0, T)$. It is assumed in above cited publications that geodesic lines generated by these coefficients are regular in a domain $Q(T) \subset \mathbb{R}^3$. The larger T is, the larger $Q(T)$ is. So, $Q(\infty) = \mathbb{R}^3$. The regularity of geodesic lines in the domain $Q(T)$ means that for any two points $x, y \in Q(T)$ there exists a single geodesic line connecting them.

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Another construction of the fundamental solution of that equation can be found in the book of Vainberg [6]. The construction of [6] imposes the non-trapping condition on variable coefficients in the principal part of the hyperbolic operator. In addition, the technique of [6] relies on the canonical Maslov operator, which is not easy to obtain explicitly. Thus, even though the structure of the fundamental solution in any of above constructions can be seen, formally at least, still many elements of this structure cannot be expressed via explicit formulas. For example, neither the solution of the eikonal equation of the method of Sobolev, nor the Maslov construction cannot be expressed via explicit formulas.

The numerical factor is important nowadays. However, it is not immediately clear how to compute numerically fundamental solutions obtained in above references. Therefore, it is also unclear how to compute solutions of Cauchy problems for heterogeneous hyperbolic equations if using those fundamental solutions.

In this paper we propose a simple method of the construction of a function, which is associated in a clear way with the fundamental solution for the acoustic equation in the 3-d case with the variable coefficient. We impose almost minimal assumptions on this coefficient. Geodesic lines are not used. It is important that our function can be both accurately and effectively approximated numerically via the Galerkin method as well as via the Finite Difference Method. Thus, we show that this function can be straightforwardly used for computations of the Cauchy problem for the heterogeneous acoustic equation.

The author was prompted to work on this paper while he was working on publications [7,8] about reconstruction procedures for phaseless inverse scattering problems. Indeed, in [7,8] the structure of the fundamental solution of the acoustic equation in time domain with the variable coefficient in its principal part is substantially used.

2. Construction

Below $x \in \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^3$ is a bounded domain. Let c_0 and c_1 be two constants such that $0 < c_0 \leq c_1$. We assume that the function $c(x)$ satisfies the following conditions

$$c \in C^1(\mathbb{R}^3), \quad c \in [c_0, c_1], \quad (1)$$

$$c(x) = 1 \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega. \quad (2)$$

Consider the following Cauchy problem

$$c(x) y_{tt} - \Delta_x y = f(x, t), \quad (x, t) \in \mathbb{R}^3 \times [0, \infty), \quad (3)$$

$$y|_{t=0} = y_t|_{t=0} = 0. \quad (4)$$

It is assumed here that f is an appropriate function such that there is a guarantee of the existence of the unique solution $y \in H^2(\mathbb{R}^3 \times (0, T))$, $\forall T > 0$ of this problem, see, e.g. Theorem 4.1 in §4 of Chapter 4 of the book of Ladyzhenskaya [9] for sufficient conditions for the latter. We will specify f later. The fundamental solution for the operator $c(x) \partial_t^2 - \Delta$ is such a function $P(x, \xi, t, \tau)$ that the solution of the problem (3), (4) can be represented in the form

$$y(x, t) = \int_0^\infty \int_{\mathbb{R}^3} P(x, \xi, t, \tau) f(\xi, \tau) d\xi d\tau. \quad (5)$$

Below we modify formula (5) and find such an analog of the function P which can be effectively computed numerically.

Let $\xi \in \mathbb{R}^3$ and $\tau \geq 0$ be parameters. Consider the following Cauchy problem

$$c(x) u_{tt} = \Delta_x u + \delta(x - \xi) \delta(t - \tau), \quad (6)$$

$$u|_{t=0} = u_t|_{t=0} = 0. \quad (7)$$

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