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Stability analysis of linear Volterra equations on time scales under bounded perturbations

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ABSTRACT

We analyze the stability of the zero solution to Volterra equations on time scales with respect to two classes of bounded perturbations. We obtain sufficient conditions on the kernel which include some known results for continuous and for discrete equations. In order to check the applicability of these conditions, we apply the theory to a test example.

1. Introduction

In this paper we consider Volterra equations on time scales of the type

$$x(t) = \delta(t) + \int_{t_0}^{t} k(t,s)x(s)\Delta s, \quad t \in [t_0, +\infty)_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}, \tag{1}$$

where $\mathbb{T}$ is a time scale, that is a nonempty, closed subset of $\mathbb{R}$. In Eq. (1) $t_0 \in \mathbb{T}$, the integral sign has to be intended as a delta-integral (see for example [8, Def. 4]) and we assume that the given real-valued functions $\delta(t)$ and $k(t,s)$ are defined in $[t_0, +\infty)_{\mathbb{T}}$ and $[t_0, +\infty)_{\mathbb{T}} \times [t_0, +\infty)_{\mathbb{T}}$ respectively.

Our starting point is [8] and we refer to it and to [2,7] for the background material concerning notations and calculus on time scales and for a survey of the existing theory about equations of the type (1). In particular, for the Δ-derivative of a function $f : \mathbb{T} \to \mathbb{R}$ (see for example [8], Def. 3) we consider the form given in [5]: $f^{\Delta}(t) = \lim_{s \to t, s \neq \sigma(t)} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$. Furthermore, we recall that, for all $t \in \mathbb{T}$ and $t < \sup \mathbb{T}$ the

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forward jump operator is given by $\sigma(t) = \{\inf \tau > t : \tau \in \mathbb{T}\}$, and that a function f is right-dense (rd) continuous if it is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim_{s \to t^-} f(s)$ exists for every left-dense point $t \in \mathbb{T}$.

Here we generalize a recent result in [8] where the Lyapunov direct method has been applied to analyze the influence of a constant perturbation on the solution of Eq. (1) over long time intervals. The classes of perturbations we are going to consider in this paper are

- $P_1 = \{\delta(t) \in BC[t_0, +\infty)_{\mathbb{T}}\}$,
- $P_2 = \{\delta(t) \in BC[t_0, +\infty)_{\mathbb{T}}$ and Δ-differentiable with $\int_0^\infty |\delta'(\tau)|\Delta\tau < \infty\}$,

where $BC[t_0, +\infty)_{\mathbb{T}}$ denotes the class of bounded continuous functions on $[t_0, +\infty)_{\mathbb{T}}$. Incidentally, P_1 and P_2 are quite popular classes of perturbations in the context of Volterra equations, both for continuous time scales and for numerical methods (see for example [3,9]). Furthermore, stability on P_1 implies stability on P_2. However, the latter class will be subject of separate investigations in this paper, because, as it will be clear in the following, within this class a slightly different analysis on the perturbation error is allowed.

2. Stability

Let P be a perturbation class for (1). We consider the following stability definition (see for example [1]).

Definition 1. The zero solution $x(t) = 0$ of the Volterra equation on time scales (1) corresponding to $\delta(t) = 0$ is called stable on $C([t_0, +\infty]_{\mathbb{T}}; \mathbb{R})$ with respect to perturbations $\delta(t) \in P$ if for each $\epsilon > 0$ there exists a $\bar{\delta} = \bar{\delta}(\epsilon, t_0) > 0$ such that $|\delta(t)| \le \bar{\delta}$ and $\delta(t) \in C([t_0, +\infty]_{\mathbb{T}}; \mathbb{R})$, implies that each solution $x(t)$ of (1) exists and satisfies $|x(t)| \le \epsilon$, for all $t \ge t_0$.

Theorem 1. *Consider the linear equation (1) and assume that k is continuous with respect to the first variable and rd-continuous with respect to the second one. If*

$$R(\sigma(t), t) := \gamma(k(\sigma(t), t)) + \int_{\sigma(t)}^{+\infty} |k^{\Delta}(\tau, t)|\Delta\tau \le 0, \tag{2}$$

and

$$\int_t^{+\infty} \left(|k(\sigma(z), z)| + \int_{t_0}^{z} |k^{\Delta}(z, \tau)|\Delta\tau \right) \Delta z \le C \tag{3}$$

$\forall t \in [t_0, +\infty)_{\mathbb{T}}$, *where $C \ge 0$, $k^{\Delta}(t, s)$ is the delta derivative of $k(t, s)$ with respect to the first variable for each fixed s, and*

$$\gamma(k(\sigma(t), t)) = \lim_{s \to t,\, s \ne \sigma(t)} \frac{|1 + (\sigma(t) - s)k(\sigma(s), s)| - 1}{\sigma(t) - s}, \tag{4}$$

the zero solution of (1) is stable with respect to the perturbation class P_1.

Proof. Let $\delta(t) \in P_1$ and consider the function

$$V(t) = |x(t) - \delta(t)| + \int_{t_0}^{t} |x(s) - \delta(s)| \int_t^{+\infty} |k^{\Delta}(\tau, s)|\Delta\tau\Delta s + \int_t^{+\infty} \left(|k(\sigma(z), z)\delta(z)| + \int_{t_0}^{z} |k^{\Delta}(z, \tau)\delta(\tau)|\Delta\tau \right) \Delta z.$$