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The profile and boundary layer for parabolic system with critical simultaneous blow-up exponent

Bingchen Liu*, Fengjie Li

College of Science, China University of Petroleum, Qingdao 266580, Shandong Province, PR China
Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN 46556, USA

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ABSTRACT

This paper deals with simultaneous blow-up solutions to a Dirichlet initial–boundary problem of the parabolic equations $u_t = \mathrm{div}(a(x)\nabla u) + \int_\Omega u^m v^s dx$ and $v_t = \mathrm{div}(b(x)\nabla v) + \int_\Omega u^q v^p dx$ in $\Omega \times [0, T)$. We complete the previous known results by covering the whole range of possible exponents. Then uniform blow-up profile is obtained for all simultaneous blow-up solutions through proving new rules for some auxiliary systems. At last, boundary layer is studied.

1. Introduction and main results

In this paper, we consider the parabolic system with variable coefficients and nonlocal sources:

$$\begin{cases} u_t(x,t) = \mathrm{div}(a(x)\nabla u(x,t)) + \displaystyle\int_\Omega u^m(x,t)v^s(x,t)dx, & (x,t) \in \Omega \times (0,T), \\ v_t(x,t) = \mathrm{div}(b(x)\nabla v(x,t)) + \displaystyle\int_\Omega u^q(x,t)v^p(x,t)dx, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, \qquad v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth open domain of $\mathbb{R}^N$; $a(x), b(x) \in C^{1+\alpha}(\bar{\Omega}, R^+), \alpha \in (0,1)$; Exponents m, s, p, $q \geq 0$; T represents the maximal existence time; Initial data u_0, v_0 are nonnegative and nontrivial, vanishing on $\partial\Omega$.

* Corresponding author at: College of Science, China University of Petroleum, Qingdao 266580, Shandong Province, PR China. Tel.: +1-574-386-0270 (Mob.).
E-mail address: bcliu@aliyun.com (B. Liu).

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The existence and uniqueness of local classical solutions and the comparison principle can be obtained by [1]. System (1.1) appears in the heat transfer of combustion and the components of the solution represent their temperatures. Recently, Wang, Song, and Lv [2] considered the blow-up time estimates of the solutions to (1.1), where different techniques are applied as for different dimension of Ω. Moreover, they obtained, if $m > q + 1$ or $p > s + 1$, there exist initial data such that non-simultaneous blow-up occurs under the conditions:

$$\nabla a(x) \cdot \nabla\varphi(x) \geq \lambda a(x)\varphi(x), \qquad \nabla b(x) \cdot \nabla\varphi(x) \geq \lambda b(x)\varphi(x) \quad \text{in } \Omega, \tag{1.2}$$

$$\operatorname{div}(a(x)\nabla u_0) + (1-\tau)\int_\Omega u_0^m v_0^s dx \geq 0, \qquad \operatorname{div}(b(x)\nabla v_0) + (1-\tau)\int_\Omega u_0^q v_0^p dx \geq 0 \quad \text{in } \Omega, \tag{1.3}$$

where τ denotes some suitable constant; $\varphi(x)$ and λ are the first eigenfunction and the corresponding eigenvalue of the operator $(-\Delta)$ in Ω with zero Dirichlet boundary, normalized by $\max_{\bar\Omega}\varphi = 1$.

Li, Huang, and Xie [3] studied the nonlocal parabolic equations $u_t = \Delta u + \int_\Omega u^m v^s dx$, $v_t = \Delta v + \int_\Omega u^q v^p dx$ in $\Omega \times (0,T)$ with zero Dirichlet boundary conditions. After obtaining the critical blow-up exponent, they proved that, if $q \geq m-1 > 0$ and $s \geq p-1 > 0$, or $q > m-1$ and $s > p-1$ and $sq > (m-1)(p-1)$, u and v blow up simultaneously for sufficiently large initial data. Moreover, the uniform blow-up profile was given about four cases: (i) $q > m-1$ and $s > p-1$ and $sq > (m-1)(p-1)$; (ii) $q > m-1 > 0$ and $s = p-1 > 0$; (iii) $q = m-1 > 0$ and $s > p-1$; (iv) $q = m-1 > 0$ and $s = p-1 > 0$.

Jiang and Li [4] studied the homogeneous Dirichlet initial–boundary problem to the equations $u_t = \Delta u + \int_\Omega e^{mu} v^s dx$, $v_t = \Delta v + \int_\Omega u^q e^{pv} dx$ in $\Omega \times (0,T)$. They obtained the uniform blow-up profile and boundary layer estimates for the cases: (i) $m > 0$ and $p > 0$; (ii) $m = p = 0$ and $qs > 1$.

Li, Liu, and Zheng [5] obtained uniform blow-up profile of solutions to the equations $u_t = \Delta u + u^m(x_0,t)v^s(x_0,t)$, $v_t = \Delta v + u^q(x_0,t)v^p(x_0,t)$ in $\Omega \times (0,T)$ with zero Dirichlet boundary conditions and any fixed point $x_0 \in \Omega$, where critical simultaneous blow-up exponent was obtained, taken of the form, $(m-1-q, p-1-s) = (0,0)$. The other studies for parabolic systems with power or exponential nonlinearities are referred to [6–19], etc. For the degenerate problems, the readers can refer to [20–23], etc.

Motivated by the above works (especially, [3–5]), we want to complete the previous known results in [2,3] by covering the whole range of possible exponents without the restriction (1.2) on $a(x)$ and $b(x)$, and then discuss the asymptotic behaviour near the blow-up time for the solutions. It is well-known that the rules founded by Souplet [24] for the auxiliary equation $u_t = \Delta u + g(t)$ in $\Omega \times (0,T)$ with $u = 0$ on $\partial\Omega \times (0,T)$ make important roles in the discussion of works [3–5], but which cannot be used directly to system (1.1) due to the different diffusion $\operatorname{div}(a(x)\nabla u)$ and $\operatorname{div}(b(x)\nabla v)$ involved in (1.1). Meanwhile, the scaling method used in [9] for the equations $u_t = \Delta u + u^m \int_\Omega v^s dx$, $v_t = \Delta v + v^p \int_\Omega u^q dx$ fails to be applied to system (1.1) also.

In the present paper, we prove some new rules for blow-up solutions to the following auxiliary parabolic problem, which are parallel to the ones in [24],

$$u_t = \operatorname{div}(a(x)\nabla u) + g_1(t), \qquad v_t = \operatorname{div}(b(x)\nabla v) + g_2(t), \quad (x,t) \in \Omega \times (0,T), \tag{1.4}$$

subject to $u = v = 0$ on $\partial\Omega \times (0,T)$. Combining with the new rules for (1.4), we construct new auxiliary functions to get rid of (1.2) according to the comparison principle, and then complete the whole classification of the exponents of (1.1). At last, we discuss uniform blow-up profile and boundary layer of solutions near the blow-up time. Assume the initial data of (1.1) satisfy

$$\operatorname{div}(a(x)\nabla u_0) + (1-\varepsilon\varphi(x))\int_\Omega u_0^m v_0^s dx, \qquad \operatorname{div}(b(x)\nabla v_0) + (1-\varepsilon\varphi(x))\int_\Omega u_0^q v_0^p dx \geq 0 \text{ in } \Omega \tag{1.5}$$

for some constant $\varepsilon \in (0,1)$. The main results of this paper are given as follows.