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An oscillation criterion for delay differential equations with several non-monotone arguments

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ABSTRACT

The oscillatory behavior of the solutions to a differential equation with several non-monotone delay arguments and non-negative coefficients is studied. A new sufficient oscillation condition, involving lim sup, is obtained. An example illustrating the significance of the result is also given.

1. Introduction

The paper deals with the differential equation with several non-monotone delay arguments of the form

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad \forall t \geq 0, \tag{1.1}$$

where p_i, $1 \leq i \leq m$, are functions of nonnegative real numbers, and τ_i, $1 \leq i \leq m$, are non-monotone functions of positive real numbers such that

$$\tau_i(t) < t, \quad t \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty, \quad 1 \leq i \leq m. \tag{1.2}$$

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Let $T_0 \in [0, +\infty)$, $\tau(t) = \min\{\tau_i(t) : i = 1, \ldots, m\}$ and $\tau_{(-1)}(t) = \sup\{s : \tau(s) \le t\}$. By a *solution* of Eq. (1.1) we understand a function $x \in C([T_0, +\infty); \mathbb{R})$, continuously differentiable on $[\tau_{(-1)}(T_0), +\infty)$ and that satisfies (1.1) for $t \ge \tau_{(-1)}(T_0)$.

A solution $x(t)$ of (1.1) is *oscillatory*, if it is neither eventually positive nor eventually negative. If there exists an eventually positive or an eventually negative solution, the equation is *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

The problem of establishing sufficient conditions for the oscillation of all solutions of Eq. (1.1) has been the subject of many investigations. See, for example, [1–15] and the references cited therein. Most of these papers concern the special case where the arguments are nondecreasing, while a small number of these papers are dealing with the general case where the arguments are non-monotone. See, for example, [1,2,15] and the references cited therein. For the general oscillation theory of differential equations the reader is referred to the monographs [16–18].

In 1978 Ladde [11] and in 1982 Ladas and Stavroulakis [10] proved that if

$$\liminf_{t\to\infty} \int_{\tau_{\max}(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{e}, \tag{1.3}$$

where $\tau_{\max}(t) = \max_{1\le i\le m}\{\tau_i(t)\}$, then all solutions of (1.1) oscillate.

In 1984, Hunt and Yorke [5] proved that if $t - \tau_i(t) \le \tau_0$, $1 \le i \le m$, and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} p_i(t)\left(t - \tau_i(t)\right) > \frac{1}{e}, \tag{1.4}$$

then all solutions of (1.1) oscillate.

When $m = 1$, that is in the special case of the equation

$$x'(t) + p(t)x\left(\tau(t)\right) = 0, \quad \forall t \ge 0, \tag{1.1$'$}$$

in 1991, Kwong [9], proved that if

$$0 < \alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \le 1/e,$$
$$\tau(t) \text{ is decreasing} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1+\ln\lambda_0}{\lambda_0}, \tag{1.5}$$

where λ_0 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$, then all solutions of $(1.1)'$ oscillate.

Recently, Braverman, Chatzarakis and Stavroulakis [1], established the following theorem in the general case that the arguments $\tau_i(t)$, $1 \le i \le m$ are non-monotone.

Theorem 1. *Assume that $p_i(t) \ge 0$, $1 \le i \le m$,*

$$h(t) = \max_{1\le i\le m} h_i(t), \quad \text{where } h_i(t) = \sup_{0\le s\le t} \tau_i(s), \quad t \ge 0 \tag{1.6}$$

and $a_r(t,s)$, $r \in \mathbb{N}$ are defined as

$$a_1(t,s) := \exp\left\{\int_s^t \sum_{i=1}^{m} p_i(\zeta)\, d\zeta\right\}, \quad a_{r+1}(t,s) := \exp\left\{\int_s^t \sum_{i=1}^{m} p_i(\zeta)a_r(\zeta, \tau_i(\zeta))\, d\zeta\right\}. \tag{1.7}$$

If for some $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r(h(t), \tau_i(\zeta)) \quad d\zeta > 1, \tag{1.8}$$