Applied Mathematics Letters 59 (2016) 109–114

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

The modified shift-splitting preconditioners for nonsymmetric saddle-point problems

Sheng-Wei Zhou[a], Ai-Li Yang[a,b,*], Yan Dou[a], Yu-Jiang Wu[a,b]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

[b] Key Laboratory of Applied Mathematics and Complex Systems, Gansu Province, Lanzhou 730000, PR China

ARTICLE INFO

Article history:
Received 29 January 2016
Received in revised form 15 March 2016
Accepted 15 March 2016
Available online 23 March 2016

Keywords:
Nonsymmetric saddle point-problem
Shift-splitting preconditioner
Iteration method
Convergence

ABSTRACT

For a nonsymmetric saddle-point problem, a modified shift-splitting (MSS) preconditioner is proposed based on a splitting of the nonsymmetric saddle-point matrix. By removing the shift term of the $(1,1)$-block of the MSS preconditioner, a local MSS (LMSS) preconditioner is also presented. Both of the two preconditioners are easy to be implemented since they have simple block structures. The convergence properties of the two iteration methods induced respectively by the MSS and the LMSS preconditioners are carefully analyzed. Numerical experiments are illustrated to show the robustness and efficiency of the MSS and the LMSS preconditioners used for accelerating the convergence of the generalized minimum residual method.

1. Introduction

We consider the following saddle-point linear system

$$\mathscr{A}u \equiv \begin{pmatrix} A & B \\ -B^T & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsymmetric positive definite matrix, $B \in \mathbb{R}^{n\times m}$ is a full column-rank matrix satisfying $m \le n$, and $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are two given vectors. The saddle-point problems frequently arise from many scientific and engineering applications, such as computational fluid dynamics, constrained optimization, and constrained least squares problem; see Refs. [1–3] for details.

In recent years, many efficient iteration methods and preconditioners used for solving saddle-point problems (1.1) have been proposed; see [2–12] and the references therein. For non-Hermitian positive definite

* Corresponding author at: School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China. Tel.: +86 931 8912483; fax: +86 931 8912481.

E-mail addresses: yangaili@lzu.edu.cn, cmalyang@139.com (A.-L. Yang).

http://dx.doi.org/10.1016/j.aml.2016.03.011
0893-9659/

system of linear equations, Bai et al. [13] proposed an efficient shift-splitting (SS) preconditioner to accelerate the convergence of the Krylov subspace methods. Then, Cao et al. in [14] employed the SS preconditioner and a local SS (LSS) preconditioner to solve symmetric saddle-point problem (1.1).

In this paper, inspired by the ideas in [13,14], we propose a modified shift-splitting (MSS) preconditioner for the nonsymmetric saddle-point problem (1.1). The convergence property of the MSS iteration method induced by the MSS preconditioner is discussed. Besides, we further introduce a local MSS (LMSS) preconditioner for the nonsymmetric saddle-point matrix. The convergence result of the corresponding LMSS iteration method is also derived. Finally, numerical experiments are presented to verify the effectiveness of the MSS and the LMSS preconditioners.

2. The modified shift-splitting preconditioner

Let $A = H + S$ be the symmetric and skew-symmetric splitting of the $(1,1)$-block of the saddle-point matrix $\mathscr{A}$, where $H = (A + A^T)/2$ and $S = (A - A^T)/2$. Similar to the shift-splitting [13,14], we propose a modified shift-splitting (MSS) for the nonsymmetric saddle-point matrix $\mathscr{A}$, i.e., $\mathscr{A} = \mathscr{P}_{\text{MSS}} - \mathscr{Q}_{\text{MSS}}$, where

$$\mathscr{P}_{\text{MSS}} = \frac{1}{2}\begin{pmatrix} \alpha I + 2H & B \\ -B^T & \alpha I \end{pmatrix} \quad \text{and} \quad \mathscr{Q}_{\text{MSS}} = \frac{1}{2}\begin{pmatrix} \alpha I - 2S & -B \\ B^T & \alpha I \end{pmatrix}. \tag{2.1}$$

Here $\alpha > 0$ is a constant and I is the identity matrix with appropriate dimension. Then, a new iteration method is defined as follows.

Method 2.1 (*The MSS Iteration Method*). *Given an initial guess* $[(x^{(0)})^T, (y^{(0)})^T]^T$, *for* $k = 0, 1, 2, \ldots,$ *until* $[(x^{(k)})^T, (y^{(k)})^T]^T$ *converges, compute*

$$\frac{1}{2}\begin{pmatrix} \alpha I + 2H & B \\ -B^T & \alpha I \end{pmatrix}\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha I - 2S & -B \\ B^T & \alpha I \end{pmatrix}\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.2}$$

The above MSS iteration can be written as the following stationary scheme:

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \mathscr{T}_\alpha \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + c, \tag{2.3}$$

where $\mathscr{T}_\alpha = \mathscr{P}_{\text{MSS}}^{-1}\mathscr{Q}_{\text{MSS}}$ is the iteration matrix and c is a certain vector. No matter using the MSS iteration scheme to approximate the solution of (1.1), or applying the MSS preconditioner $\mathscr{P}_{\text{MSS}}$ to accelerate the convergence of Krylov subspace methods, we need to solve a system of linear equations $\mathscr{P}_{\text{MSS}} z = r$ at each step of iteration methods. This system of linear equations can be solved as follows.

Algorithm 2.1. *For a given vector* $r = [r_1^T, r_2^T]^T$, *we compute vector* $z = [z_1^T, z_2^T]^T$ *according to the following steps*:

(1) *solve* $(\alpha I + 2H + (1/\alpha)BB^T)z_1 = 2r_1 - (2/\alpha)Br_2$;
(2) $z_2 = (1/\alpha)(2r_2 + B^T z_1)$.

From Algorithm 2.1 we only need to solve a linear subsystem when solving $\mathscr{P}_{\text{MSS}} z = r$. Since the coefficient matrix of this linear system is symmetric positive-definite for any $\alpha > 0$, we can employ the conjugate gradient (CG) or the preconditioned CG (PCG) method to approximate its solution.

In the following, we discuss the convergence property of the MSS iteration scheme (2.3). Denote by $\rho(\mathscr{T}_\alpha)$ the spectral radius of iteration matrix $\mathscr{T}_\alpha$. Then the MSS iteration method is convergent if and only if $\rho(\mathscr{T}_\alpha) < 1$. Before giving the convergence result, we first give some useful lemmas.