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Boundedness in a multi-dimensional chemotaxis–haptotaxis model with nonlinear diffusion

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ABSTRACT

We consider the chemotaxis–haptotaxis model

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1-u-w), \\ \quad x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, \quad x \in \Omega, t > 0, \\ w_t = -vw, \quad x \in \Omega, t > 0 \end{cases}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^n (n \geq 2)$, where χ, ξ and μ are positive parameters, and the diffusivity $D(u)$ is assumed to generalize the prototype $D(u) = \delta(u+1)^{-\alpha}$ with $\alpha \in \mathbb{R}$. Under zero-flux boundary conditions, it is shown that for sufficiently smooth initial data (u_0, v_0, w_0) and $\alpha < \frac{2-n}{n+2}$, the corresponding initial–boundary problem possesses a unique global-in-time classical solution which is uniformly bounded, which improves the previous results.

1. Introduction

In recent years, mathematical models of cancer have been developed to describe many facets of the tumor expansion and its metastases, aiming to shed fresh impetus on tumor formation, progression and treatment. Although the specific mechanisms underlying the migration of cancer cells into surrounding tissue remain sketchy, it is recognized that diffusion and taxis are two of the many mechanisms of cancer cell motility. The term taxis characterizes the directed migration of cells along concentration gradients of some signal available in the environment, and usually refers to chemotaxis (the directed cell motion in response to concentration gradients of some chemoattractant/chemorepellent) or haptotaxis (the directed migratory response of tumor cells to the extracellular environment). In [1,2], Chaplain and Lolas described cancer

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invasion into surrounding healthy tissue by the chemotaxis–haptotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1-u-w), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0, \\ D(u)\dfrac{\partial u}{\partial \nu} - \chi u \dfrac{\partial v}{\partial \nu} - \xi u \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$, the three variables u, v and w represent the cancer cell density, the diffusible matrix-degrading enzyme (MDE) concentration and the density of a static tissue referred to as extracellular matrix (ECM), $D(u)$ describes the density-dependent motility of cancer cells through the ECM, χ and ξ measure the chemotactic and haptotactic sensitivities, respectively.

Recently, (1.1) and its analogue have received much attention [3–9]. Compared with the chemotaxis-only system, the analysis of (1.1) has to cope with new mathematical challenges. The analytic obstacle essentially stems from the fact that apparently the chemotaxis and haptotaxis terms in the first equation of (1.1) require different mathematical treatment. As far as we know, few results are available for the case of the density dependence of the self-diffusion of cancer cells. Indeed, the global existence of solutions to (1.1) was even proved in [10] under the assumption that either $n \leq 8$ and $\alpha < \frac{4-n^2}{n^2+4n}$, or $n \geq 9$ and $\alpha < (\sqrt{8n(n+1)} - n^2 - n - 2)/(n^2+2n)$, and when $\alpha < \frac{2-n}{n}$, the global boundedness of solutions to (1.1) was obtained in the recent paper [7].

The purpose of this paper is to replace the condition $\alpha < \frac{2-n}{n}$ in [7] by $\alpha < \frac{2-n}{n+2}$. Accordingly, we henceforth assume that for some $\vartheta \in (0,1)$

$$\begin{cases} u_0 \in C(\bar{\Omega}) \quad \text{with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{with } v_0 \geq 0 \text{ in } \Omega, \\ w_0 \in C^{2+\vartheta}(\bar{\Omega}) \quad \text{with } w_0 \geq 0 \text{ in } \bar{\Omega} \text{ and } \dfrac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

and for some $\alpha \in \mathbb{R}$ and $\delta > 0$

$$D \in C^2([0,\infty)), \qquad D(u) \geq \delta(u+1)^{-\alpha} \quad \text{for all } u \geq 0. \tag{1.3}$$

Our main results in this paper are stated as follows.

Theorem 1.1. *Let $n \geq 2, \chi > 0, \xi > 0$, $\mu > 0$, and let D be a function satisfying (1.3) with $\alpha < \frac{2-n}{n+2}$. Then for any initial data fulfilling (1.2), problem (1.1) admits a unique classical solution which is globally bounded in $\Omega \times (0,\infty)$.*

2. Proof of Theorem 1.1

The following lemma provides the basic estimates of solutions of (1.1).

Lemma 2.1 (*Lemma 3.1 of [8]*). *Let (u,v,w) be the solution of (1.1). Then there exists $C > 0$ depending on $n, \|v_0\|_{W^{1,2}(\Omega)}$ and $\|u_0\|_{L^1(\Omega)}$ such that*

$$\|u(\cdot,t)\|_{L^1(\Omega)} \leq C, \qquad \|v(\cdot,t)\|_{L^1(\Omega)} \leq C, \qquad \|\nabla v(\cdot,t)\|_{L^2(\Omega)} \leq C \quad \text{for all } t \in (0, T_{max}). \tag{2.1}$$

Lemma 2.2 (*Lemma 3.2 of [7]*). *Let (u,v,w) be the classical solution of (1.1) in $\Omega \times (0,T_{max})$. Then for any $k > 1$,*

$$-\int_\Omega (u+1)^{k-1} \nabla \cdot (u \nabla w) \leq c_1 \left(\int_\Omega (u+1)^k + \int_\Omega (u+1)^k v + k \int_\Omega (u+1)^{k-1} |\nabla u| \right) \tag{2.2}$$

with constant $c_1 > 0$ independent of k.