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The fourth order elliptic problem with exponential nonlinearity

Baishun Lai[a,*], Qing Luo[b]

[a] Institute of Contemporary Mathematics, Henan University, Kaifeng 475004, PR China
[b] School of Mathematics and Statistics, Henan University, Kaifeng 475004, PR China

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ABSTRACT

We consider the entire solution of the semilinear biharmonic equation

$$\Delta^2 u = e^u, \quad \text{in } \mathbb{R}^N,\ N \geq 1.$$

For $N = 3$, we obtain the asymptotic of the entire nonradial solution which extends the results of Lai and Ye (in press). Besides, a new singular solution of the above equation is constructed by derivative estimates for $N \geq 13$.

1. Introduction

In the present paper, we are interested in the entire solution of the semilinear biharmonic equation

$$\Delta^2 u = e^u, \quad \text{in } \mathbb{R}^N,\ N \geq 1. \tag{1.1}$$

Recently, the fourth order equations with an exponential non-linearity have attracted the interest of many researchers. In particular, lots of efforts have been devoted to existence/multiplicity, stability, qualitative properties of these solutions of (1.1). In the "conformal dimension" $N = 4$, problem (1.1), which is called Liouville's equation, is invariant under the change of the conformal transformation, this equation also appears naturally in conformal geometry as the constant Q-curvature problem. The existence and the asymptotic of solutions with finite total curvature, i.e. $e^u \in L^1(\mathbb{R}^4)$ were first studied in [1–3]. Subsequently, the similar results were well established for "supercritical dimensions" $N \geq 5$ (see [4–7]). More recently, the stability properties of the entire solution for (1.1) was considered by [8], and studied in depth in [9,10]. The nonexistence of radial entire solution of (1.1) for $N = 2$ is proved by Berchio-Farina-Ferrero-Gazzola [8], and this result is extended to nonradial case by Lai–Ye [11].

* Corresponding author.
E-mail addresses: laibaishun@henu.edu.cn (B. Lai), lq@henu.edu.cn (Q. Luo).

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For convenience, Now we give the radial version of (1.1) as follows

$$\begin{cases} \Delta^2 u_{\alpha,\beta}(r) = \lambda \exp(u_{\alpha,\beta}(r)) \quad \text{for } r \in [0, R(\alpha,\beta)), \\ u_{\alpha,\beta}(0) = \alpha, \qquad \Delta u_{\alpha,\beta}(0) = \beta, \qquad u'_{\alpha,\beta}(0) = (\Delta u_{\alpha,\beta})'(0) = 0, \end{cases} \tag{1.2}$$

where $[0, R(\alpha,\beta))$ is the maximal interval of existence.

To illuminate the motivations of this paper in detail, we now only recall some results about the asymptotic and existence of the radial solution of (1.2), being of relevance for the present paper.

Theorem 1.1. *For $N \geq 3$, local solutions to (1.2) satisfy*

$$u_{\alpha,\beta}(r) \geq \alpha + \frac{\beta}{2N} r^2 \quad \text{for all } r \in [0, R(\alpha,\beta)). \tag{1.3}$$

Furthermore, for any $\alpha \in \mathbb{R}$ there exists $\beta_0 \in [-4Ne^{\frac{\alpha}{2}}, 0)$ such that

(a) *if $\beta < \beta_0$, then $R(\alpha,\beta) = +\infty$ and in addition to (1.3), one has the upper bound*

$$u_{\alpha,\beta}(r) \leq \alpha - \frac{\beta_0 - \beta}{2N} r^2 \quad \text{for all } r \in [0, \infty).$$

(b) *If $\beta = \beta_0$, then the solution u_{α,β_0}, which is called separatrix, such that for $r \to \infty$*

$(b)_1$ *if $N = 3$, then*

$$u_{\alpha,\beta_0}(r) = a_1 r + \frac{1}{2}\int_0^\infty t^3 e^{u_{\alpha,\beta_0}} dt - \frac{1}{6r}\int_0^\infty t^4 e^{u_{\alpha,\beta_0}} dt + O(e^{-cr}) \quad \text{with some } c > 0,$$

where

$$a_1 = -\frac{1}{4\pi}\int_{\mathbb{R}^3} e^{u_{\alpha,\beta_0}} dx;$$

$(b)_2$ *if $N = 4$, then*

$$u_{\alpha,\beta_0} = \alpha - 4\log\left(1 + \frac{e^{\frac{\alpha}{2}}}{8\sqrt{6}} r^2\right);$$

$(b)_3$ *if $N \geq 5$, then*

$$\lim_{r\to+\infty}(u_{\alpha,\beta_0} + 4\log r) = \log 8(N-2)(N-4).$$

(c) *For any $\beta \leq \beta_0$, $\lim_{r\to\infty} \Delta u_{\alpha,\beta}(r) \in \mathbb{R}_-$ and $\lim_{r\to\infty} \Delta u_{\alpha,\beta}(r) = 0$ if and only if $\beta = \beta_0$.*

Arioli, Gazzola and Grunau [4] proved (1.3)(a), $(b)_3$ of the above results (or see [8]); Lin [2] obtained $(b)_2$. Besides, $(b)_1$ and (c) were proved by Lai–Ye [11].

For convenience, we now need the following notations which will be used throughout the paper. Set

$$\overline{f} := \fint_{\partial\mathbb{B}_r(0)} f d\sigma = \frac{1}{|\partial\mathbb{B}_r(0)|}\int_{\partial\mathbb{B}_r(0)} f d\sigma,$$

and denote $f \backsim g$ if there are two positive constants C_1, C_2 such that $C_1 f \leq g \leq C_2 f$.

The first purpose of the present paper is to extend some above results to the nonradial case:

Theorem 1.2. *Let $N = 3$ and u be the solution of (1.1). Then*

(i) *There exists two positive constants β_1, β_2 such that*

$$-\beta_1 r^2 \leq \bar{u}(r) \leq -\beta_2 r \quad \text{for } r > 1.$$