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An improved error bound on Gauss quadrature[☆]

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ABSTRACT

In this paper, the refined estimates on aliasing errors on integration of Chebyshev polynomials by Gauss quadrature are present, which, together with the asymptotic formulae on the coefficients of Chebyshev expansions, deduces an improved convergence rate for Gauss quadrature. The convergence orders are attainable for some functions of finite regularities.

1. Introduction

The computation of integrals of the form of

$$I[f] = \int_{-1}^{1} f(x)dx \tag{1.1}$$

is one of oldest and most important issues in numerical analysis. Among all interpolation type quadrature rules with n nodes, the Gauss quadrature

$$I_n^G[f] = \sum_{j=1}^{n} w_j f(x_j) \tag{1.2}$$

has the highest accuracy of degree $2n-1$, where x_k are the zeros of the Legendre polynomial of degree n, ordered by $-1 < x_1 < x_2 < \cdots < x_n < 1$, and w_k the corresponding weights in the n-point Gauss quadrature ($k = 1, 2, \ldots, n$). Fast evaluation of the nodes and weights with $O(n)$ operations for the Gauss quadrature was given by Glaser, Liu and Rokhlin [1], Bogaert, Michiels and Fostier [2], and Hale and Townsend [3]. A Matlab file for computation of these nodes and weights can be found in Chebfun system [4].

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By using new asymptotics on the coefficients of Chebyshev expansions for functions of finite regularities, Trefethen in [5,6] showed that for an integer $k \geq 1$, if $f(x)$ has an absolutely continuous $(k-1)$st derivative $f^{(k-1)}$ on $[-1,1]$ and a kth derivative $f^{(k)}$ of bounded variation $V_k = \mathrm{Var}(f^{(k)}) < \infty$, then for each $j \geq k+1$,

$$|a_j| \leq \frac{2V_k}{\pi j(j-1)\cdots(j-k)}, \tag{1.3}$$

and for all $n \geq k/2+1$

$$|E_n^G[f]| = |I[f] - I_n^G[f]| \leq \frac{32V_k}{15k\pi(2n-1-k)^k}, \tag{1.4}$$

where a_j is the coefficient of the following Chebyshev series expansion

$$f(x) = \sum_{j=0}^{\infty}{}' a_j T_j(x). \tag{1.5}$$

Here the prime denotes the summation whose first term is halved, $T_j(x) = \cos(j\cos^{-1}x)$ denotes the Chebyshev polynomial of degree j, and the Chebyshev coefficient a_j is defined by

$$a_j = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x)T_j(x)}{\sqrt{1-x^2}}dx, \quad j = 0, 1, \ldots. \tag{1.6}$$

One further power of n for n-point Gauss quadrature is given in Xiang and Bornemann [7] for $f \in X^s$ with $s \geq 2$, based on the work of Curtis and Rabinowitz [8] and Riess and Johnson [9] from the early 1970s, and a refined estimate for Gauss quadrature is applied to Chebyshev polynomials due to Petras in 1995 [10]. Here, we say $f \in X^s$ if the Chebyshev coefficient a_j satisfies that $a_j = O(j^{-s-1})$ [7]. From [5,6], we see that if $f(x)$ has an absolutely continuous $(k-1)$st derivative $f^{(k-1)}$ on $[-1,1]$ (if $k \geq 1$) and $V_k < \infty$ then $f \in X^k$.

In this paper, we will present refined estimates on the aliasing errors about the integration of Chebyshev polynomials by Gauss quadrature, and give an improved convergence rate for n-point Gauss quadrature for $f \in X^s$.

2. On the optimal general convergence rates for Gauss quadrature

From (1.5), we see that the quadrature of Gauss-quadrature can be written as

$$E_n^G[f] = \sum_{j=2n}^{\infty} a_j E_n^G[T_j], \quad f \in X^s,\ s > 0. \tag{2.1}$$

Knowledge of the errors in the numerical integration of Chebyshev polynomials of the first kind, $T_j(x)$, is useful in various situations including the error in integrating functions of low-order continuity or with branch-point singularities and computing the norm of the error functional of the given rule in a certain family of Hilbert spaces of analytic functions [8].

Applying the estimate in Gatteschi [11]

$$x_j = \cos\left(\phi_j + \frac{1}{8}\cot(\phi_j) + O(j^{-2}n^{-1})\right), \quad 1 \leq j \leq \frac{n}{2},\ \phi_j = \frac{(4j-1)\pi}{4n+2}$$

and an $O(n^{-1})$ bound on the weights w_j, Curtis and Rabinowitz [8] showed that the error in integrating the Chebyshev polynomials satisfies

$$E_n^G[Tm] = \begin{cases} (-1)^j\dfrac{2}{4r^2-1} + O\left(\dfrac{m^2}{n^3}\right) + O\left(\dfrac{m\log n}{n^2}\right), & -n < r < n, \\ (-1)^j\dfrac{\pi}{2} + O\left(\dfrac{m^2}{n^3}\right) + O\left(\dfrac{m\log n}{n^2}\right), & r = \pm n \end{cases}$$

for $2n \leq m = j(4n+2) + 2r$ with $-n \leq r \leq n$ and $j \geq 0$.