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Uniqueness and counterexamples in some inverse source problems[☆]

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ABSTRACT

Uniqueness of a solution is investigated for some inverse source problems arising in linear parabolic equations. We prove new uniqueness results formulated in Theorems 3.1 and 3.2. We also show optimality of the conditions under which uniqueness holds by explicitly constructing counterexamples, that is by constructing more than one solution in the case when the conditions for uniqueness are violated.

1. Introduction

Mathematical models related to inverse source problems (ISP's) for parabolic equations arise in various applications such as in the location of a pollutant source in groundwater flow and in the design and control of various heat processes. Partly due to its importance in applications there has been a recent surge on research for various types of ISP's for parabolic problems, see for example references in [1,2].

A classical uniqueness result, [3], asserts that for a heat (or diffusion) process with coefficients solely depending on the space variables, a time-independent heat source can be uniquely reconstructed from final time data. It is straightforward to see that there cannot be any uniqueness result for this type of inverse problem in the class of general heat sources that depends on both space and time. It is, however, not immediately clear in terms of uniqueness of a spacewise dependent heat source from final time data in the case when the heat or diffusion process has time-dependent coefficients. That particular case is researched in [2].

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In the present work, we shall investigate the given conditions for uniqueness derived in [2], and construct examples showing that when these conditions are violated there is more than one solution. Moreover, we generalize the results in [2], holding for spacewise dependent heat sources, to a wider class of sources being the product of a known time-dependent part and an unknown (to be reconstructed) spacewise dependent part. Sources of the latter form have been used, for example, in [4].

2. Optimality of the conditions for uniqueness derived in Ref. [2]

Consider a bounded domain Ω with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. To set out to investigate conditions for uniqueness, we start by being more precise and recall that the following inverse source problem (ISP) is studied in [2]: Find a solution pair $\{u(x,t), f(x)\}$ such that

$$\begin{cases} \partial_t u + \nabla \cdot (-\boldsymbol{A}(x,t)\nabla u) + c(x,t)u = f(x) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

given the final time data

$$u(x,T) = \psi_T(x), \quad \text{for } x \in \Omega \tag{2}$$

where $T > 0$. It is assumed that

$$\boldsymbol{A}(x,t) = \left(a_{i,j}(x,t)\right)_{i,j=1,\dots,n}, \quad \boldsymbol{A} = \boldsymbol{A}^t,$$

together with the requirement

$$\boldsymbol{\xi}^t \cdot \boldsymbol{A}\boldsymbol{\xi} = \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \geq C|\boldsymbol{\xi}|^2 \tag{3}$$

for every x, t and any $\boldsymbol{\xi}^t = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, where $n \geq 1$.

Uniqueness in the retrieval of the solution and source to the ISP (1)–(2) is proved in [2, Thm. 3.7] under the conditions that

$$\boldsymbol{\xi}^t \cdot \partial_t \left(\boldsymbol{A}^{\frac{1}{2}}\right) \boldsymbol{A}^{\frac{1}{2}}\boldsymbol{\xi} \leq 0, \quad \forall \xi \in \mathbb{R}^n \quad \text{and} \quad \partial_t c(t) \leq 0 \quad \forall t \in [0,T]. \tag{4}$$

Note that the matrix $\boldsymbol{A}$ obeys (3), and is therefore positive definite in space and time, thus there exists a unique positive definite square root $\boldsymbol{A}^{\frac{1}{2}}$, cf. [5, Chpt. VII, §3]. For scalar coefficients, the conditions (4) translate into the requirements that both $A(t)$ and $c(t)$ decrease in time.

It is natural to ponder on the question whether the requirement to have the coefficients decreasing in time can be relaxed or perhaps dropped altogether and still have uniqueness of a solution to the ISP (1)–(2). Rather surprisingly, the following example shows that without the condition of decay in time we cannot guarantee uniqueness in the above ISP.

Example 1. Consider the following one-dimensional ISP for $x, t \in (0, \pi)$ having zero final time data,

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + c(t)u(x,t) = f(x), \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = 0, \\ u(x,\pi) = 0, \end{cases} \tag{5}$$

with the choice $c(t) = \frac{1-\cos(t)}{\sin(t)} - 1$. Easy calculations show that the coefficient c can be naturally extended to $t = 0$ with $c(0) = -1$, and that $c'(t) = (\cos(t) + 1)^{-1} > 0$. Hence, c is smooth and non-decreasing for t