Applied Mathematics Letters 58 (2016) 62–68

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

# Logarithmic improvement of regularity criteria for the Navier–Stokes equations in terms of pressure

Chuong V. Tran[a], Xinwei Yu[b,*]

[a] *School of Mathematics and Statistics, University of St. Andrews, St Andrews KY16 9SS, United Kingdom*
[b] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada*

ARTICLE INFO

*Article history:*
Received 15 December 2015
Received in revised form 9 February 2016
Accepted 9 February 2016
Available online 17 February 2016

*Keywords:*
Navier–Stokes
Global regularity
Pressure
Prodi–Serrin
Multiplier spaces
Logarithmic improvement

ABSTRACT

In this article we prove a logarithmic improvement of regularity criteria in the multiplier spaces for the Cauchy problem of the incompressible Navier–Stokes equations in terms of pressure. This improves the main result in Benbernou (2009).


## 1. Introduction

At the center stage of mathematical fluid mechanics are the incompressible Navier–Stokes equations

$$u_t + u \cdot \nabla u = -\nabla p + \nu \triangle u, \qquad (x,t) \in \Omega \times (0,\infty) \tag{1}$$

$$\operatorname{div} u = 0, \qquad (x,t) \in \Omega \times (0,\infty) \tag{2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega \tag{3}$$

with appropriate boundary conditions. Here $\Omega \subseteq \mathbb{R}^d$ is a domain with certain regularity, $u : \Omega \mapsto \mathbb{R}^d$ is the velocity field, $p : \Omega \mapsto \mathbb{R}$ is the pressure, and $\nu > 0$ is the (dimensionless) viscosity. The system (1)–(3) on one hand describes the motion of viscous Newtonian fluids, while on the other hand serves as the starting point of mathematical modeling of many other types of fluids, such as non-Newtonian fluids, magnetic fluids, electric fluids, and ferro-fluids. In this article we focus on the Cauchy problem of (1)–(3), where $\Omega = \mathbb{R}^d$.

* Corresponding author.
*E-mail addresses:* chuong@mcs.st-and.ac.uk (C.V. Tran), xinwei2@ualberta.ca (X. Yu).

http://dx.doi.org/10.1016/j.aml.2016.02.003
0893-9659/

As (1)–(3) serve as the foundation of the modern quantitative theory of incompressible fluids, it is important to have complete mathematical understanding of these equations. However the achievement of this goal is still out of the question. In particular, there is still no satisfactory answer to the question of well-posedness of the Cauchy problem of (1)–(3).

The first systematic study of this well-posedness problem (for the case $d = 3$) was carried out by Jean Leray in [1], where it is shown that for arbitrary $T \in (0, \infty]$ there is at least one function $u(x, t)$ satisfying the following:

i. $u \in L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d))$;
ii. $u$ satisfies (1) and (2) in the sense of distributions;
iii. $u$ takes the initial value in the $L^2$ sense: $\lim_{t\searrow 0} \|u(\cdot, t) - u_0(\cdot)\|_{L^2} = 0$;
iv. $u$ satisfies the energy inequality

$$\|u(\cdot,t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\cdot,\tau)\|_{L^2}^2 \mathrm{d}\tau \leqslant \|u_0\|_{L^2}^2 \tag{4}$$

for all $0 \leqslant t \leqslant T$.

Such a function $u(x, t)$ is called a Leray–Hopf weak solution for (1)–(3) in $\mathbb{R}^d \times [0, T)$.

It is easy to show that if a Leray-Hopf weak solution is smooth, then it satisfies (1)–(3) in the classical sense. One could further show that such a smooth Leray-Hopf solution must be unique. Therefore the well-posedness problem would be settled if all Leray-Hopf solutions could be shown to be smooth. However such a general result has not been established up to now. On the other hand, various additional assumptions guaranteeing the smoothness of Leray-Hopf solutions have been discovered. For example, it has been shown that if a Leray-Hopf solution $u(x, t)$ further satisfies

$$u \in L^r(0, T; L^s(\mathbb{R}^d)) \quad \text{with } \frac{2}{r} + \frac{d}{s} \leqslant 1,\ d < s \leqslant \infty, \tag{5}$$

then $u(x, t)$ is smooth and is thus a classical solution, see e.g. [2–4]. The borderline case $u \in L^\infty(0, T; L^3)$ is much more complicated and requires a totally different approach. It was settled much later by Escauriaza, Seregin, and Sverak in [5]. Many generalizations and refinements of (5) have been proved, see e.g. [6–11].

If we formally take the divergence of (1) we obtain the following relation between $u$ and $p$:

$$-\triangle p = \mathrm{div}(\mathrm{div}(u \otimes u)) \tag{6}$$

where $u \otimes u$ is a $d \times d$ matrix with $i$-$j$ entry $u_i u_j$. Thus intuitively we have $p \sim u^2$. Transforming (5) via this relation, we expect that

$$p \in L^r(0, T; L^s(\mathbb{R}^d)) \quad \text{with } \frac{2}{r} + \frac{d}{s} \leqslant 2,\ \frac{d}{2} < s \leqslant \infty \tag{7}$$

should guarantee the smoothness of $u$. This is indeed the case and was confirmed in [12,13].

Many efforts have been made to refine (7), see e.g. [14–21]. It is worth mentioning that the relation (6) has also played crucial roles in the proofs of other regularity criteria not of the Prodi-Serrin type. For example, in [22] it is used to show that Leray-Hopf weak solutions are regular as long as either $|u|^2 + 2p$ is bounded above or $p$ is bounded below. Among generalizations of (7), it is shown in [14] that $u$ is smooth as long as $p \in L^{2/(2-r)}(0, T; \dot{X}_r(\mathbb{R}^d)^d)$ for $0 < r \leqslant 1$ where $\dot{X}_r(\mathbb{R}^d)$ is the multiplier space. Multiplier spaces are defined for $0 \leqslant r < d/2$ and functions $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ through the norm

$$\|f\|_{\dot{X}_r} := \sup_{\|g\|_{\dot{H}^r} \leqslant 1} \|fg\|_{L^2} < \infty, \tag{8}$$