Applied Mathematics Letters 58 (2016) 69–73

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Research announcement

Optimal strategies for asset allocation and consumption under stochastic volatility

Qiang Zhang*, Lei Ge

Department of Mathematics, City University of Hong Kong, Hong Kong

ARTICLE INFO

Article history:
Received 5 February 2016
Accepted 5 February 2016
Available online 18 February 2016

Keywords:
Portfolio selection
Optimal strategies
Consumption
Stochastic volatility
Utility maximization

ABSTRACT

Selecting optimal asset allocation and consumption strategies is an important, but difficult, topic in modern finance. The dynamics is governed by a nonlinear partial differential equation. Stochastic volatility adds further complication. Even to obtain a numerical solution is challenging. Here, we develop a closed-form approximate solution. We show that our theoretical predictions for the optimal asset allocation strategy and the optimal consumption strategy are in surprisingly good agreement with the results from full numerical computations.

1. Introduction

Empirical studies suggest that the volatility should be modeled as a stochastic process. Thus the study of portfolio selection under a stochastic volatility becomes a very important topic in modern finance. The works on portfolio selection under stochastic volatility without consumption can be found in [1–5] and the references therein. Works on portfolio selection under stochastic volatility with consumption are also available in the literature. Fleming and Hernández-Hernández studied the case when the volatility is governed by two independent Brownian motions [6]. Chacko and Viceira developed an approximate solution for the case of an infinite investment horizon and obtained exact solutions for certain special cases [7]. Noh and Kim developed a perturbation solution for the consumption strategy in an infinite horizon [8]. Liu studied this problem in a general setting and derived the solutions for certain special cases [9]. In practical applications, one needs to consider portfolio optimization with consumption in a finite time horizon. However, to obtain numerical solutions for this problem is difficult and complicated. In this letter, we develop closed-form approximate solutions for the optimal asset allocation strategy and the optimal consumption strategy under maximization of expected power utility. We show that our theoretical predictions are in surprisingly good agreement with the exact numerical solutions.

* Corresponding author.
E-mail addresses: mazq@cityu.edu.hk (Q. Zhang), leige365@outlook.com (L. Ge).

http://dx.doi.org/10.1016/j.aml.2016.02.005

2. Governing equations

We consider a market which consists of a riskless asset B and a risky asset S. B and S are governed by

$$dB = rBdt \quad \text{and} \quad dS_t = S_t\big(\mu(v_t) + \sqrt{v_t}dW_t^S\big),$$

where r is a constant risk-free return rate, v_t is the stochastic variance of S and $\mu(v_t)$ is the return of S. Following Merton [10] and Heston [11], we assume that $A = \frac{\mu - r}{v}$ is a constant. v_t is governed by

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^v,$$

where dW_t^v and dW_t^S are the increments of Wiener processes and have a constant correlation ρ, θ is the long run average volatility, κ is the rate at which v_t reverts to θ and ξ is the variance of $\sqrt{v_t}$. The parameters κ, θ, ξ are positive constants and need to satisfy the Feller condition, $2\kappa\theta \geq \xi^2$, to ensure that v_t is strictly positive.

An investor with an initial wealth w_0 has the constant relative risk aversion (CRRA) utility function given by $u(x) = (1-\gamma)^{-1}x^{1-\gamma}$, where γ is a constant with $\gamma > 0$. The investor's objective is to maximize

$$\sup_{\phi, c} E\left[\int_0^T \alpha e^{-\beta t}u(c_t)dt + (1-\alpha)e^{-\beta T}u(w_T)\right], \tag{2.1}$$

where $\phi_t w_t$ is the wealth allocated to the risky asset S, $(1-\phi_t)w_t$ is the wealth allocated to the riskless asset B, c_t is the consumption rate at time t, E is the expectation and β is the subjective discount rate. α measures the relative importance of the intermediate consumption and the terminal wealth. The wealth process is

$$dw_t = \phi_t w_t S_t^{-1}dS_t + (1-\phi_t)w_t r dt - c_t dt.$$

We will follow the Hamilton–Jacobi–Bellman (H–J–B) dynamic programming approach to solve this optimization problem. Let V be the value function. By the conjecture $V(t,w,v) = e^{-\beta t}\frac{w^{1-\gamma}}{1-\gamma}f(v,t)^\gamma$, the H–J–B equation for f is [9]

$$-f_\tau + a_1 v f_{vv} + (a_2 v + a_3)f_v + a_4 v\frac{f_v^2}{f} + (a_5 v + a_6)f + \alpha^{\frac{1}{\gamma}} = 0 \tag{2.2}$$

with $f(\tau = 0, v) = (1-\alpha)^{\frac{1}{\gamma}}$, where $\tau = T - t$ and

$$a_1 = \xi^2/2, \qquad a_2 = (1-\gamma)\gamma^{-1}\rho A\xi - \kappa, \qquad a_3 = \kappa\theta, \qquad a_4 = (\gamma-1)\xi^2(1-\rho^2)/2,$$
$$a_5 = (1-\gamma)\gamma^{-2}A^2/2, \qquad a_6 = (1-\gamma)\gamma^{-1}r - \beta\gamma^{-1} \quad \text{and} \quad a_7 = \xi^2(\gamma + \rho^2 - \gamma\rho^2)/2.$$

The strategies that optimize Eq. (2.1) are

$$c^\star = \alpha^{\frac{1}{\gamma}}wf^{-1} \quad \text{and} \quad \phi^\star = A\gamma^{-1} + \rho\xi f^{-1}(f)_v. \tag{2.3}$$

Liu derived closed-form solutions of Eq. (2.2) for two special cases: no consumption ($\alpha = 0$) or dW_t^S and dW_t^v are perfectly correlated or anti-correlated ($\rho = \pm 1$) [9]. In this letter, we consider general values of α and ρ, namely, $0 \leq \alpha \leq 1$ and $-1 \leq \rho \leq 1$. Unfortunately, there is no general solution of Eqs. (2.2) and (2.3). Numerical computation has been the only available method to determine the solution.

3. Our closed-form approximate solution

The difficulty in obtaining a closed-form solution for f is rooted from the coexistence of two factors. The first factor is the inhomogeneous term $\alpha^{\frac{1}{\gamma}}$, and the second one is the high-order derivative and the nonlinearity, namely, the terms $a_1 v f_{vv}$ and $a_4 v\frac{f_v^2}{f}$ appearing in Eq. (2.2). Our approach is to deal with