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Research announcement

Uniqueness of the solution to inverse obstacle scattering with non-over-determined data

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ABSTRACT

It is proved that the scattering amplitude $A(\beta, \alpha_0, k_0)$, known for all $\beta \in S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$, $\alpha_0 \in S^2$ is fixed, $k_0 > 0$ is fixed, determines the surface S of the obstacle and the boundary condition on S uniquely. The boundary condition on S is either the Dirichlet, or Neumann, or the impedance one. The uniqueness theorems for the solution of inverse scattering problems with non-over-determined data were not known for many decades. Such a theorem is proved in this paper for inverse scattering by obstacles for the first time.

1. Introduction

The uniqueness theorems for the solution of inverse scattering problems with non-over-determined data were not known for many decades. Such a theorem is proved in this paper for inverse scattering by obstacles for the first time. In [1–3] such theorems are proved for the first time for inverse scattering by potentials.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected C^2-smooth boundary S, $D' := \mathbb{R}^3 \setminus D$ be the unbounded exterior domain and S^2 be the unit sphere in $\mathbb{R}^3$.

Consider the scattering problem:

$$(\nabla^2 + k^2)u = 0 \quad in\ D', \quad \Gamma_j u|_S = 0, \qquad u = e^{ik\alpha\cdot x} + v, \tag{1}$$

where $k > 0$ is a constant, $\alpha \in S^2$ is a unit vector in the direction of the propagation of the incident plane wave $e^{ik\alpha\cdot x}$, S^2 is the unit sphere in $\mathbb{R}^3$, $\Gamma_1 u := u$, $\Gamma_2 u := u_N$, N is the unit normal to S pointing out of D, u_N is the normal derivative of u, $\Gamma_3 u := u_N + hu$, $h = const$, Im$h \geq 0$, h is the boundary impedance, and the scattered field v satisfies the radiation condition

$$v_r - ikv = o\Big(\frac{1}{r}\Big), \quad r := |x| \to \infty. \tag{2}$$

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The scattering amplitude $A(\beta,\alpha,k)$ is defined by the following formula:

$$v = A(\beta,\alpha,k)\frac{e^{ikr}}{r} + o\Big(\frac{1}{r}\Big), \quad r := |x| \to \infty, \ \frac{x}{r} = \beta, \tag{3}$$

where $\alpha,\beta \in S^2$, β is the direction of the scattered wave, α is the direction of the incident wave. For a bounded domain $o(\frac{1}{r}) = O(\frac{1}{r^2})$ in formula (3). The function $A(\beta,\alpha,k)$, the scattering amplitude, can be measured experimentally. Let us call it the scattering data. It is known (see [4], p. 25) that the solution to the scattering problem (1)–(3) does exist and is unique.

The *inverse scattering problem (IP)* consists of finding S and the boundary condition on S from the scattering data. It was first proved by M. Schiffer in the sixties of the last century that if the boundary condition is the Dirichlet one, then the surface S is uniquely determined by the scattering data $A(\beta,\alpha_0,k)$ known for a fixed $\alpha = \alpha_0$ all $\beta \in S^2$ and all $k \in (a,b)$, $0 \le a < b$. M. Schiffer's proof was not published by him. This proof can be found in [4], p. 85, and the acknowledgement of M. Schiffer's contribution is on p. 399 in [4].

A.G. Ramm was the first to prove that the scattering data $A(\beta,\alpha,k_0)$, known for all β in a solid angle, all α in a solid angle and a fixed $k = k_0 > 0$, determine uniquely the boundary S *and the boundary condition.* This condition was assumed of one of the three types Γ_j, $j = 1,2$ or 3, (see [4], Chapter 2, for the proof of these results). By subindex zero fixed values of the parameters are denoted, for example, k_0, α_0. By solid angle in this paper an open subset of S^2 is understood.

In [4], p. 62, it is proved that for smooth bounded obstacles the scattering amplitude $A(\beta,\alpha,k)$ is an analytic function of β and α on the analytic variety $M := \{z|z \in \mathbb{C}^3, z\cdot z = 1\}$, where $z \cdot z := \sum_{m=1}^{3} z_m^2$. The unit sphere S^2 is a subset of M. If $A(\beta,\alpha,k)$ as a function of β is known on an open subset of S^2, it is uniquely extended to all of S^2 (and to all of M) by analyticity. The same is true if $A(\beta,\alpha,k)$ as a function of α is known on an open subset of S^2.

In papers [5] and [6] a new approach to a proof of the uniqueness theorems for inverse obstacle scattering problem (IP) was given. This approach is used in our paper.

Since the sixties of the last century the uniqueness theorem for IP with *non-over-determined data* $A(\beta) := A(\beta,\alpha_0,k_0)$ was not known. In paper [7] the uniqueness theorem for IP with such data was proved for strictly convex smooth obstacles. The proof in [7] was based on the results concerning location of resonances for a pair of such obstacles. These results are technically difficult to obtain and they hold for two strictly convex obstacles with a positive distance between them.

The purpose of this paper is to prove the uniqueness theorem for IP without any convexity assumption about S. By the boundary condition any of the three conditions Γ_j are understood below.

Theorem 1. *The surface S and the boundary condition on S are uniquely determined by the data $A(\beta)$ known in a solid angle.*

In Section 2 some auxiliary material is formulated in five lemmas and Theorem 1 is proved.

2. Proof of Theorem 1

First we give an auxiliary material. It consists of five lemmas which are proved by the author, except for Lemma 3, which was known (it was proved first by V. Kupradze in 1934, then by H. Freudenthal in 1938, then by I. Vekua in 1943 and by F. Rellich in 1943, see references in the monograph [4], p. 397). This lemma is often referred to as Rellich's lemma. A proof of it, based on a new idea, is given in paper [8].

Denote by $G(x,y,k)$ the Green's function corresponding to the scattering problem (1)–(3). The parameter $k > 0$ is assumed fixed in what follows. For definiteness we assume below the Dirichlet boundary condition,