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Continuity of the eigenvalues of nonhomogeneous hinged vibrating rods

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ABSTRACT

In this paper we will prove that the eigenvalues of nonhomogeneous hinged vibrating rods have a strongly continuous dependence on weights, i.e., as nonlinear functionals of weights, eigenvalues are continuous in weights with respect to the weak topologies in the Lebesgue spaces $\mathcal{L}^p$.

1. Introduction

Motivated by extremal problems of weighted eigenvalues, we will prove in this paper that the eigenvalues of nonhomogeneous hinged vibrating rods have a strongly continuous dependence on weights, i.e., as nonlinear functionals of weights, eigenvalues are continuous in weights with respect to the weak topologies in the Lebesgue spaces $\mathcal{L}^p$.

For $1 \le p \le \infty$, let $\mathcal{L}^p := L^p([0,1],\mathbb{R})$ be the Lebesgue space with the $\mathcal{L}^p$ norm denoted by $\|\cdot\|_p = \|\cdot\|_{L^p[0,1]}$. Denote

$$\mathcal{L}^p_+ := \left\{\mu \in \mathcal{L}^p : \mu(x) \ge 0 \text{ a.e. } x \in [0,1], \text{ and } \int_0^1 \mu(x)dx > 0\right\}.$$

Let $\rho \in \mathcal{L}^p_+$, called a weight. We are concerned with the eigenvalues of nonhomogeneous hinged vibrating rods

$$y^{(4)}(x) - \lambda\rho(x)y(x) = 0, \quad x \in [0,1], \tag{1.1}$$

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with the boundary condition

$$y(0) = y(1) = 0 = y''(0) = y''(1). \tag{1.2}$$

It is well-known that problem (1.1)–(1.2) has a sequence of (real) eigenvalues

$$0 < \lambda_1(\rho) < \lambda_2(\rho) < \cdots < \lambda_m(\rho) < \cdots$$

such that $\lim_{m\to\infty} \lambda_m(\rho) = +\infty$, see [1].

We say that $\rho_n \to \rho$ in $(\mathcal{L}^p, w_p)$, if

$$\int_0^1 \rho_n v \, \mathrm{d}x \to \int_0^1 \rho v \, \mathrm{d}x \quad \forall v \in \mathcal{L}^{p^*},$$

where $p^* = p/(p-1)$ is the conjugate exponent of p. A functional $f : \mathcal{L}^p \to \mathbb{R}$ is said to be strongly continuous if $f : (\mathcal{L}^p, w_p) \to \mathbb{R}$ is continuous. Evidently, strong continuity of f implies that $f : (\mathcal{L}^p, \|\cdot\|_p) \to \mathbb{R}$ is continuous.

The main result of this paper is the following strong continuity of $\lambda_m(\rho)$ in ρ.

Theorem 1.1. *For each $m \in \mathbb{N}$, as a nonlinear functional, $\lambda_m(\rho)$ is strongly continuous in $\rho \in \mathcal{L}^p_+$, where $1 \le p \le \infty$.*

In papers [2–5], the authors used the argument method to show that eigenvalues of the second order operators have a strongly continuous dependence on potentials. These strong continuity results have been applied to solve several interesting extremal problems and optimal estimations for the corresponding eigenvalues in papers [6,7]. In this paper, we will study the dependence of eigenvalues of the fourth order equation by the variational characterization of eigenvalues, which is a totally different approach from the case of the second order equation. Based on the continuity results of this paper, we will study some minimization problems of the corresponding eigenvalues in the further work.

2. Preliminary results

Given $\rho \in \mathcal{L}^p_+$, where $1 \le p \le \infty$, and $\lambda \in \mathbb{R}$. Let $\varphi_i(x, \lambda, \rho)$ be the fundamental solution of Eq. (1.1) satisfying

$$(y(0), y'(0), y''(0), y'''(0))^T = e_i,$$

where $1 \le i \le 4$. Results in [2,5] show that solutions of (1.1) have strongly continuous dependence on weights ρ.

Lemma 2.1. *As nonlinear operators, the following solution mappings*

$$\mathbb{R} \times (\mathcal{L}^p, w_p) \to (C^3, \|\cdot\|_{C^3}), \qquad (\lambda, \rho) \to \varphi_i(\cdot, \lambda, \rho), \tag{2.1}$$

are continuous, where $1 \le i \le 4$. Here $C^3 := C^3([0,1], \mathbb{R})$.

As for the first eigenvalue $\lambda_1(\rho)$, one has the following minimization characterization.

Lemma 2.2 (*[1]*). *There holds*

$$\lambda_1(\rho) = \min_{\substack{u \in C_0^2 \\ u \ne 0}} \frac{\int_0^1 (u'')^2 \, \mathrm{d}x}{\int_0^1 \rho u^2 \, \mathrm{d}x}, \tag{2.2}$$

where

$$C_0^2 := \left\{ u \in C^2([0,1], \mathbb{R}) : u(0) = u(1) = u''(0) = u''(1) = 0 \right\}.$$