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Interfacial phenomena of the forward backward convection–diffusion equations

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ABSTRACT

This paper is devoted to the interfacial phenomena of a class of forward backward convection–diffusion equations. Under the assumption that the equations have classical solutions, we prove that the forward region expands with a positive rate in one dimension.

1. Introduction

In this paper, we consider the following forward backward convection–diffusion equation:

$$u_t = \operatorname{div}\left(\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) + F(x, \nabla u), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open bounded domain and $T > 0$. The structural assumptions on the odd function ϕ are:

$$\phi(s) \in C^\infty(\mathbb{R}), \qquad \phi''(s) > 0, \quad \forall s \in (-\alpha, \alpha); \qquad \phi''(s) < 0, \quad \forall |s| > \alpha. \tag{2}$$

The non-monotonicity of ϕ' leads to a forward backward partial differential equation (1) of parabolic type. A typical example of the forward backward parabolic equation was considered by Höllig [1]. Similar equations to (1) with conditions (2) can also be found in fluid mechanics [2] and image processing [3].

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There are many other forward backward parabolic equations arising from biological modeling. Anguige et al. [4], Bao and Zhou [5], Turchin [6] and Horstmann [7] all constructed the following model:

$$u_t = (D(u)u_x)_x + g(u), \tag{3}$$

where $u \in [0,1]$, and $D(u)$ have positive and negative values corresponding to different physical interpretations.

Recently, Zhang [8], Kim and Yan [9,10] proved the existence of infinitely many Lipschitz weak solutions for the forward backward parabolic equation from image processing. Wang et al. [11] investigated the properties of the Young measure solutions of the following forward backward convection–diffusion equation:

$$u_t = \text{div } \Phi(\nabla u) + \text{div } A(x,t,u) + B(x,t,u), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{4}$$

where the monotonicity condition

$$(\Phi(\xi) - \Phi(\zeta)) \cdot (\xi - \zeta) \geq 0, \tag{5}$$

is violated for some $\xi, \zeta \in \mathbb{R}^n$.

Classical solutions of (1) without convection terms (so that solutions are at least C^1) have been investigated in the last decade. Kawohl and Kutev [12] proved that global-in-time classical solutions exist if the initial conditions are in the forward region, while it is remarked in [13] that local-in-time classical solutions cannot exist unless the initial conditions are at least C^∞ in its backward region.

In the framework of classical solutions, we denote

$$Q^+ := \{(x,t) \in Q_T \text{ with } u_x \in (-\alpha, \alpha)\} \tag{6}$$

the forward (subsonic) region of Eq. (1),

$$Q^- := \{(x,t) \in Q_T \text{ with } u_x \in \mathbb{R} \setminus [-\alpha, \alpha]\} \tag{7}$$

the backward (supersonic) region and

$$Q^0 := \{(x,t) \in Q_T \text{ with } |u_x| = \alpha\} \tag{8}$$

the degenerate (sonic) region. Note in the sonic region, Eq. (1) is a first order equation. These notions are borrowed from the theory of transonic flow for an ideal gas, which is modeled by an elliptic–hyperbolic operator of divergence type [12].

When one considers the interfacial behaviors of the Perona–Malik equation from image processing, under the framework of the classical solution, in the one dimensional case Kawohl and Kutev [12] proved the shrinking property of the union of supersonic and sonic regions as time increasing and the supersonic region exists for all $t > 0$ under some structural assumptions that the solution exists. Furthermore, Ghisi and Gobbino [14] proved similar results for one dimensional case and expanding property of supersonic regions in high dimensional case.

This paper is organized as follows. In Section 2 we first state a result that is related to the free boundary problem of the degenerate parabolic equations which will be used in our proof of the main theorem. Second, we state our results for the forward backward equation with convection term and the related free boundary problem in one dimension.

2. Main results

For simplicity, we assume that ϕ satisfies condition (2) with $\phi'''(\alpha) < 0$,

$$F(x,y) \in C^\infty(\mathbb{R}^2) \quad \text{and} \quad F_x(x,\alpha) = F_x(x,-\alpha) = A < 0. \tag{9}$$

In order to prove our main theorem, we first state a result in [14] for a free boundary problem involving degenerate parabolic equations.