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Convexity of the support of the displacement interpolation: Counterexamples

Filippo Santambrogio[a,*], Xu-Jia Wang[b]

[a] *Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay Cedex, France*

[b] *Centre for Mathematics and its Applications, Australian National University, ACT 0200, Australia*

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ABSTRACT

Given two smooth and positive densities ρ_0, ρ_1 on two compact convex sets K_0, K_1, respectively, we consider the question whether the support of the measure ρ_t obtained as the geodesic interpolant of ρ_0 and ρ_1 in the Wasserstein space $\mathbb{W}_2(\mathbb{R}^d)$ is necessarily convex or not. We prove that this is not the case, even when ρ_0 and ρ_1 are uniform measures.

1. Introduction

One of the main features of optimal transport theory (we refer to [1] and [2] for a general presentation) is the fact that it provides an original and efficient way to define interpolations between probability measures.

Given a domain $\Omega \subset \mathbb{R}^d$ (that we take compact for simplicity and convex for the sake of the interpolation), we define the space $\mathbb{W}_2(\Omega)$ as the space of probabilities on Ω endowed with the distance W_2, defined through

$$W_2^2(\mu,\nu) = \min\left\{\int_{\Omega\times\Omega} |x-y|^2 \,\mathrm{d}\gamma \;:\; \gamma \in \Pi(\mu,\nu)\right\},$$

where $\Pi(\mu,\nu)$ is the set of the so-called *transport plans*, i.e.

$$\Pi(\mu,\nu) = \{\gamma \in \mathcal{P}(\Omega\times\Omega) : (\pi_x)_\#\gamma = \mu,\ (\pi_y)_\#\gamma = \nu\},$$

where $\pi_x(x,y) := x$ and $\pi_y(x,y) = y$ are the standard projections on the two factors of $\Omega \times \Omega$.

* Corresponding author.
E-mail addresses: filippo.santambrogio@math.u-psud.fr (F. Santambrogio), Xu-Jia.Wang@anu.edu.au (X.-J. Wang).

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It is possible to prove that the above minimization problem has a solution, which is unique and of the form $\gamma = (id, T)_{\#}\mu$ (i.e. it is concentrated on the graph of a map $T : \Omega \to \Omega$, called *optimal transport map*) in many situations (in particular if $\mu \ll \mathcal{L}^d$) and that W_2, i.e. the square root of the minimal value above, is indeed a distance on $\mathcal{P}(\Omega)$.

The space $\mathbb{W}_2(\Omega)$ can be checked to be a geodesic space, where for $\mu, \nu \in \mathcal{P}(\Omega)$ the (unique) geodesic curve connecting them is obtained through

$$\rho_t := ((1-t)\,\mathrm{id} + tT)_{\#}\mu, \tag{1.1}$$

where T is the optimal transport map from μ to ν. This provides a useful interpolation between μ and ν which is very different from the linear interpolation $(1-t)\mu + t\nu$. In particular, it is useful in many applications, for instance in image processing: this is the case when doing histograms interpolations (roughly speaking, the average between a distribution of pixel colors which are almost white and another were they are almost black should be a distribution with intermediate gray pixels, and not one with half pixels which are white and half which are black), or when artificially creating intermediate images between two pictures representing a same object which has moved (and the goal is to find the same object in the middle, instead of two half objects at the starting and arrival spots). But this very interpolation has also a lot of mathematical applications, as it was first pointed out by McCann in [3]. Indeed, McCann found a class of functionals $F : \mathcal{P}(\Omega) \to \mathbb{R}$ which are convex along these geodesic lines (but not necessarily convex in the usual sense, think for instance at $\mu \mapsto \int\int |x-y|^2 \mathrm{d}\mu(x)\mathrm{d}\mu(y)$), thus making possible to obtain uniqueness results or sufficient optimality conditions (see [4], for instance) for variational problems involving them. Also, this notion of convexity, called *displacement convexity* (the above interpolation is also called *displacement interpolation*) is an important notion in the study of gradient flows of these functionals (see [5]). We also recall that the interpolation ρ_t can be found numerically via one of the most classical algorithm for optimal transport, the so-called Benamou–Brenier method [6]. By solving the kinetic energy minimization problem

$$\min\left\{\int_0^1\int_\Omega |v_t|^2 \mathrm{d}\rho_t\,\mathrm{d}t \;:\; \partial_t\rho_t + \nabla\cdot(v_t\rho_t) = 0\right\},$$

among curves of measures with given initial and final data, one recovers the above interpolation, and the optimal velocity field v_t allows to find T (as we have $v_t(x) = (T - \mathrm{id})\circ(T_t)^{-1}$ with $T_t := (1-t)\,\mathrm{id} + tT$).

Finally, let us remark that the interpolated measures ρ_t can also be considered as weighted barycenters between ρ_0 and ρ_1, as they solve the minimization problem

$$\min\left\{(1-t)W_2^2(\rho,\rho_0) + tW_2^2(\rho,\rho_1) \;:\; \rho \in \mathcal{P}(\Omega)\right\},$$

a minimization problem which has ρ_t as unique solution (provided one of the measures ρ_0, ρ_1 is absolutely continuous). This problem has also been considered when more than two measures are given, in order to find the weighted barycenter of many of them, solving (see [7])

$$\min\left\{\sum_{i=1}^n \lambda_i W_2^2(\rho,\rho_i) \;:\; \rho \in \mathcal{P}(\Omega)\right\} \tag{1.2}$$

(a minimization problem that can be also recast in the setting of multi-marginal problems, as in [8]). Note that when $n \geq 3$ this variational definition of the barycenter is the only possible one, differently from the case $n = 2$ where one can simply use the geodesic in $\mathbb{W}_2$.

In optimal transport theory, and in particular for regularity issues, the convexity of the support of a measure is an important fact, and some results are only available under this assumption. In particular, the optimal transport map between two smooth densities ρ_0, ρ_1 is smooth itself, provided their supports are convex (this theory was first developed by Caffarelli, [9–11], see [12] for a survey). Since it is possible to write optimality conditions for the minimizers of (1.2) in terms of the optimal maps T_i sending the optimal