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Alternating-directional PMHSS iteration method for a class of two-by-two block linear systems[☆]

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ABSTRACT

In Bai et al. (2013), a preconditioned modified HSS (PMHSS) method was proposed for a class of two-by-two block systems of linear equations. In this paper, the PMHSS method is modified by adding one more parameter in the iteration. Convergence of the modified PMHSS method is guaranteed. Theoretic analysis and numerical experiment show that the modification improves the PMHSS method.

1. Introduction

We consider the two-by-two block systems of linear equations of the form

$$Ax \equiv \begin{pmatrix} W & -T \\ T & W \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \equiv g, \tag{1}$$

where $W, T \in \mathbb{R}^{n \times n}$ are symmetric and positive semidefinite matrices with at least one of them being positive definite. Without loss of generality, throughout the paper, we assume that in (1), W is symmetric positive definite and T is symmetric positive semidefinite. This class of linear systems can be regarded as a special case of generalized saddle point problem [1,2]. It arises from finite element discretizations of elliptic partial differential equation, constrained optimization problems such as distributed control problems [3–6] and from real equivalent formulations of complex symmetric linear systems [7].

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The system of linear equations (1) can be solved by the preconditioned GMRES method, or can be first symmetrized and then solved by the preconditioned MINRES method. Bai et al. proposed a class of preconditioned modified Hermitian and skew-Hermitian splitting iteration methods (PMHSS), which can be simply described as follows (see [8,9]).

The PMHSS iteration method [10].

Let $(y^{(0)^T}, z^{(0)^T})^T \in \mathbb{R}^{2n}$ be an initial guess, with $y^{(0)}, z^{(0)} \in \mathbb{R}^n$. For $k = 0, 1, 2, \ldots$, until the sequence of iterates $\{(y^{(k)^T}, z^{(k)^T})^T\}_{k=0}^{\infty} \subset \mathbb{R}^{2n}$ converges, compute the next iterate $(y^{(k+1)^T}, z^{(k+1)^T})^T$ according to the following procedure:

$$\begin{cases} \begin{pmatrix} \alpha V + W & 0 \\ 0 & \alpha V + W \end{pmatrix} \begin{pmatrix} y^{(k+\frac{1}{2})} \\ z^{(k+\frac{1}{2})} \end{pmatrix} = \begin{pmatrix} \alpha V & T \\ -T & \alpha V \end{pmatrix} \begin{pmatrix} y^{(k)} \\ z^{(k)} \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}, \\ \begin{pmatrix} \alpha V + T & 0 \\ 0 & \alpha V + T \end{pmatrix} \begin{pmatrix} y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = \begin{pmatrix} \alpha V & -W \\ W & \alpha V \end{pmatrix} \begin{pmatrix} y^{(k+\frac{1}{2})} \\ z^{(k+\frac{1}{2})} \end{pmatrix} + \begin{pmatrix} q \\ -p \end{pmatrix}, \end{cases} \tag{2}$$

where α is a given positive constant and V is a prescribed symmetric positive definite matrix.

The iteration matrix of the PMHSS method is

$$L(V;\alpha) = \begin{pmatrix} \alpha V + T & 0 \\ 0 & \alpha V + T \end{pmatrix}^{-1} \begin{pmatrix} \alpha V & -W \\ W & \alpha V \end{pmatrix} \begin{pmatrix} \alpha V + W & 0 \\ 0 & \alpha V + W \end{pmatrix}^{-1} \begin{pmatrix} \alpha V & T \\ -T & \alpha V \end{pmatrix}. \tag{3}$$

Let $\lambda(C)$, $\rho(C)$ denote the spectral set, spectral radius of the matrix C, respectively and

$$\sigma(\alpha) = \max_{\widetilde{\lambda}_j \in sp(V^{-1}W)} \frac{\sqrt{\alpha^2 + \widetilde{\lambda}_j^2}}{\alpha + \widetilde{\lambda}_j} \cdot \max_{\widetilde{\mu}_j \in sp(V^{-1}T)} \frac{\sqrt{\alpha^2 + \widetilde{\mu}_j^2}}{\alpha + \widetilde{\mu}_j}. \tag{4}$$

Then for the PMHSS iteration, we have the following convergence theorem (see [10]).

Theorem 1.1. *The spectral radius of the PMHSS iteration matrix $L(V;\alpha)$ satisfies*

$$\rho(L(V;\alpha)) \leqslant \sigma(\alpha) \leqslant \max_{\widetilde{\lambda}_j \in sp(V^{-1}W)} \frac{\sqrt{\alpha^2 + \widetilde{\lambda}_j^2}}{\alpha + \widetilde{\lambda}_j} < 1, \quad \forall\, \alpha > 0,$$

i.e., the PMHSS iteration method converges unconditionally to the unique solution of the two-by-two block linear system (1) for any initial guess.

From the PMHSS method and Theorem 1.1, we see that in the PMHSS iteration only one parameter α can be chosen, and its optimal choice is just related to the matrix W, one of the two matrices W and T contained in the system matrix A. This seems arbitrary and shows that the PMHSS iteration has room to improve. One situation is when W and T differ in weight, for example, our assumption: W is positive definite and T is only positive semidefinite. But since the PMHSS is blind to any difference between W and T, it still picks one parameter. Conceivably, if W and T could be treated differently with regard to their own characteristics, better algorithms would be possible. This is the motivational thought that drives our study in this paper. Specifically, we will propose a modified PMHSS iteration method—the alternating-directional PMHSS (ADPMHSS), which includes two parameters that can be tailored to reflect each individual characteristic of W and T, and consequently ADPMHSS converges at least as fast as PMHSS and can be faster when added parameter is suitably chosen.

The rest of this paper is organized as follows. In Section 2, we propose the ADPMHSS method and develop a convergence theory for it. In Section 3, we give some numerical examples to show the effectiveness of the ADPMHSS method. Section 4 is a simple remark.