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Smooth solution of a nonlocal Fokker–Planck equation associated with stochastic systems with Lévy noise

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ABSTRACT

It is shown that the solution of a nonlocal Fokker–Planck equation is smooth with respect to both time and space variable whenever the divergence of the smooth drift has a lower bound.

1. Introduction

We consider the following nonlocal Fokker–Planck equation defined on $\mathbf{R}^n$

$$\begin{cases} u_t + \Lambda^\alpha u + \nabla \cdot (\boldsymbol{a}(x)u) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\boldsymbol{a} : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a time independent function (called 'drift'). The fractional Laplacian $\Lambda^\alpha, \alpha \in (0,2)$, is defined by

$$\Lambda^\alpha f(x) = c_{\alpha,n} P.V. \int_{\mathbf{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}} dy, \tag{1.2}$$

where $c_{\alpha,n}$ is a constant depending only on n and α.

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If $\boldsymbol{a}(x)$ is bounded, the existence and regularity of solutions for (1.1) was studied in [1]. Moreover, if $\boldsymbol{a}(x)$ belongs to some Kato class, the heat kernel of the semigroup generated by the operator $\Lambda^\alpha + \nabla \cdot (\boldsymbol{a}\cdot)$ was obtained in [2,3]. In these works, the drift $\boldsymbol{a}(x)$ is required to satisfy the following condition

$$\sup_{x\in\mathbf{R}^n} \int_{B(x,1)} |\boldsymbol{a}(x)| dx < \infty, \tag{1.3}$$

where $B(x,r)$ denotes the ball centered at x with radius r.

Eq. (1.1) is the Fokker–Planck for a stochastic differential equation with a random source denoted by $\widehat{X_t}$ and a drift term given by a deterministic function $\boldsymbol{a}(x)$:

$$dX_t = \boldsymbol{a}(X_t)dt + d\widehat{X_t}, \tag{1.4}$$

where $\widehat{X_t}$ is the α-stable Lévy process, and the solution of (1.1) is the probability density of X_t, see e.g. [4,5]. Some important drifts, such as Ornstein–Uhlenbeck drift $\boldsymbol{a}(x) = -\boldsymbol{x}$ and double well drift $\boldsymbol{a}(x) = x - x^3$ in dimension 1, do not belong to the class determined by (1.3). Thus, it is natural to consider Eq. (1.1) with drifts growing at infinity.

In [6], Xie et al. showed that the solution of Eq. (1.1) is smooth in the case $\boldsymbol{a}(x) = -\boldsymbol{x}$. The proof relies on the following formula of the solution

$$u(t,x) = \int_{\mathbf{R}^n} e^{nt} K\left(\frac{1-e^{-\alpha t}}{\alpha}, e^{-t}x, y\right) u_0(y)dy,$$

where $K(t,x,y)$ is the integral kernel of the heat semigroup $e^{-t\Lambda^\alpha}$ (see [7]).

However, no precise presentation of the solution for Eq. (1.1) with general drift $\boldsymbol{a}(x)$ is available yet. In this paper, we overcome the difficulty to show the solution is smooth for a class of smooth drifts.

Theorem 1.1. *Assume that* $0 < \alpha < 2, u_0 \in L^2(\mathbf{R}^n)$, $\boldsymbol{a}(x) \in C^\infty(\mathbf{R}^n)$ *and* $div\boldsymbol{a}(x) \geq c$ *for some constant* $c \in \mathbf{R}$. *Then Eq.* (1.1) *has a unique solution* $u \in C^\infty((0,\infty)\times\mathbf{R}^n)$.

We give some remarks on Theorem 1.1. In dimension 1, Theorem 1.1 holds if $\boldsymbol{a}(x) = x^3 - x$. Also, the solution of (1.1) is shown to be Hölder continuous if the drift is Hölder continuous, see e.g. [8–10]. Our proof is different from these works. Finally, the restriction $div\boldsymbol{a}(x) \geq c$ is only used in the existence of solution. Whether it can be removed is an open problem.

2. Two commutator estimates

Let $b(x,\xi)$ be a continuous function on $\mathbf{R}^n \times \mathbf{R}^n$. Define the pseudo-differential operator

$$b(x,D)f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} b(x,\xi)\widehat{f}(\xi)d\xi, \quad f \in \mathscr{S}(\mathbf{R}^n),$$

where $D = \frac{1}{i}(\partial_{x_1},\ldots,\partial_{x_n})$, $\widehat{f}$ is the Fourier transform given by $\widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx$, $\mathscr{S}(\mathbf{R}^n)$ is the Schwartz class. We call $b(x,\xi)$ the symbol of $b(x,D)$. In particular, let $b(x,\xi) = |\xi|^\alpha$ and $(1+|\xi|^2)^{s/2}$, we obtain Λ^α and $J_s = (1-\triangle)^{s/2}$ $(s\in\mathbf{R})$, respectively. The Sobolev spaces $H^s(\mathbf{R}^n)$ are defined as the completion of Schwartz space with respect to the norm $\|f\|_{H^s} = \|J_s f\|_{L^2}$. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$, it is easy to see that for all $s \in \mathbf{R}$

$$\|\varphi f\|_{H^s} \leq C\|f\|_{H^s}. \tag{2.5}$$

Let $m \in \mathbf{R}$. We say a function $b(x,\xi)$ belongs to S^m if for all μ_1,μ_2

$$|\partial_\xi^{\mu_1}\partial_x^{\mu_2} b(x,\xi)| \leq C_{\mu_1,\mu_2}(1+|\xi|)^{m-|\mu_1|}, \quad x,\xi \in \mathbf{R}^n.$$

We recall the following important properties of S^m, see e.g. [11, p. 251].