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Revisit to Fritz John's paper on the blow-up of nonlinear wave equations[☆]

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ABSTRACT

In Fritz John's famous paper (1979), he discovered that for the wave equation $\Box u = |u|^p$, where $1 < p < 1 + \sqrt{2}$ and $\Box$ denoting the d'Alembertian, there is no global solution for any nontrivial and compactly supported initial data. This paper is intended to simplify his proof by applying a Gronwall's type inequality.

1. Introduction

In 1979, Fritz John published his pioneering work [1], which was the first one that discovered the critical power of the blow-up phenomenon for wave equations. After this article, many people worked on this kind of blow-up problem. For details and many other related references, see [2–13]. These work generalize the critical power to other dimensions, some of them also provide simpler proof by imposing additional assumptions or by applying different methods.

The paper [1] claims the following well-known Theorem.

Theorem 1.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies*

$$\phi(0) = 0, \quad \limsup_{s\to 0} \phi(s)/|s| < \infty.$$

☆ The original paper is "Blow-up of solutions of nonlinear wave equations in three space dimensions, Manusctipta Math. 28 (1979) 235–268". See [1].

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Moreover, suppose there exists $A > 0$ *and* $1 < p < 1 + \sqrt{2}$ *such that for all* $s \in \mathbb{R}, \phi(s) \geq A|s|^p$. *Then for any function* $u \in C^2\big(\mathbb{R}^3 \times [0, \infty)\big)$ *that solves*

$$\begin{cases} \square u(x,t) = \phi\big(u(x,t)\big) & \text{for } x \in \mathbb{R}^3,\ t \in (0,\infty), \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}^3, \\ u_t(x,0) = g(x) & \text{for } x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $f \in C^3(\mathbb{R}^3), g \in C^2(\mathbb{R}^3)$ *and both of them have compact support, we have* $u \equiv 0$ *in* $\mathbb{R}^3 \times [0, \infty)$. *Here* $\square$ *denotes the d'Alembertian operator:* $\square = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$.

The key step to prove this Theorem is the statement as following.

Theorem 1.2. *Let* $A > 0$, $1 < p < 1 + \sqrt{2}$, *let* u *be a* $C^2\big(\mathbb{R}^3 \times [0, \infty)\big)$ *solution of* $\square u \geq A|u|^p$. *Moreover, suppose there exists a point* $(x^0, t_0) \in \mathbb{R}^4$ *such that* $u^0(x,t) \geq 0$ *for* $(x,t) \in \Gamma^+(x^0, t_0)$, *then* u *has compact support and supp* $u \subset \Gamma^-(x^0, t_0)$.

Remark 1.1. If u satisfies $\square u = w$ with initial data f and g, then one decomposes u by $u = u^0 + u^1$, where u^0 solves

$$\begin{cases} \square u^0(x,t) = 0 & \text{for } x \in \mathbb{R}^3,\ t \in (0,\infty), \\ u^0(x,0) = f(x) & \text{for } x \in \mathbb{R}^3, \\ u^0_t(x,0) = g(x) & \text{for } x \in \mathbb{R}^3, \end{cases} \tag{2}$$

and u^1 suffices

$$\begin{cases} \square u^1(x,t) = w(x,t) & \text{for } x \in \mathbb{R}^3,\ t \in (0,\infty), \\ u^1(x,0) = u^1_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{3}$$

Remark 1.2. For any $(x^0, t_0) \in \mathbb{R}^4$, the forward cone $\Gamma^+(x^0, t_0)$ and the backward cone $\Gamma^-(x^0, t_0)$ are defined as $\Gamma^+(x^0, t_0) = \{(x,t) : |x - x^0| \leq t - t_0, t \geq 0\}$ and $\Gamma^-(x^0, t_0) = \{(x,t) : |x - x^0| \leq t_0 - t, t \geq 0\}$.

In the proof of Theorem 1.2, [1] employs a technical induction which requires complicated calculations. By introducing a suitable nonlinear functional, this paper gives a much more succinct proof which follows a Gronwall's type inequality.

The organization of this paper is as following: In Section 2, it is shown how Theorem 1.2 implies Theorem 1.1, the argument is from [1]. In addition, some notations and a basic Lemma are introduced, where the Lemma is the key technique used in Section 3.2. In Section 3, we prove Theorem 1.2. More precisely, Section 3.1, the first part of the proof, follows from [1] with modifications while Section 3.2, the rest part of the proof, comes from our own observation.

2. Preliminaries

2.1. Theorem 1.2 implies Theorem 1.1

Proof. Firstly, one can assume that both f and g have support in $B(\mathbf{0}, \rho) \triangleq \{x \in \mathbb{R}^3 : |x| < \rho\}$, then by Huygens' principle, $u^0 \equiv 0$ in $\Gamma^+(\mathbf{0}, \rho)$. It follows from Theorem 1.2 that supp $u \subset \Gamma^-(\mathbf{0}, \rho)$. Secondly, one considers the function $v(x,t) \triangleq u(x, \rho - t)$ for $x \in \mathbb{R}^3, 0 \leq t \leq \rho$. Using the assumptions on ϕ in Theorem 1.1, one can see $|\square v| \leq M|v|$ for some fixed M depending on u and ϕ. Then by energy estimate, $v \equiv 0$ in $\Gamma^-(\mathbf{0}, \rho)$. Thus Theorem 1.1 is verified. $\square$