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A vector generalization of Volterra type differential–difference equations

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ABSTRACT

A discrete $(N+1)\times(N+1)$ matrix spectral problem and the corresponding hierarchy of Volterra type differential–difference equations are proposed. It is also shown that the hierarchy of differential–difference equations possesses the Hamiltonian structures. Infinite conservation laws for the first nontrivial member in the hierarchy are given.

1. Introduction

The differential–difference equations, which are always used to model some physical phenomena in mathematical physics, plasma physics, optical physics, biology, etc. [1–4], have attracted more and more attention in recent years. A well-known fact is that a hierarchy of integrable differential–difference equations can be generated through the isospectral compatibility condition for a pair of discrete spectral problems. Many integrable differential–difference equations have been presented and studied systematically such as the Toda lattice, the Volterra model, the discrete modified KdV equation, the discrete nonlinear Schrödinger equation, etc. However, the class of discrete models which are integrable by the Lax pairs is few as compared with that of continuous models [5–7], and they are almost all derived from discrete 2×2 or 3×3 matrix spectral problems [8–14]. While for higher-order discrete matrix spectral problems, owing to their complexity, the relevant research is fewer and not satisfactory. Therefore, it is necessary to make further research on differential–difference equations derived from the higher-order discrete matrix spectral problems [15–17]. It is well-known that the existence of Hamiltonian structures is a very important feature for integrability of

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soliton systems. The trace identity [18] provides a powerful tool for constructing Hamiltonian structures of differential–difference equations, from which the Hamiltonian structures of many soliton equations are obtained. Furthermore, the conservation laws [19] also play important roles in discussing the integrability for soliton equations. From the viewpoint of physical and numerical analysis, it is also very interesting to know whether there exist conservation laws for a lattice system. Generally, the infinitely many conservation laws can be obtained from the Lax pair, the Bäcklund transformation, from the formal solutions of eigenfunctions, or from the trace identity [20–23]. In Ref. [8], a discrete 2×2 matrix spectral problem and the corresponding hierarchy of Volterra type differential–difference equations were proposed. It is shown that the hierarchy of differential–difference equations possesses the Hamiltonian structures. A Darboux transformation for the discrete spectral problem is found. As an application, two-soliton solutions for the first system of differential–difference equations in the hierarchy are given.

In this letter, we propose a discrete $(N+1) \times (N+1)$ matrix spectral problem with $2N$ potentials and derive a vector generalization of Volterra type differential–difference equations. The trace identity is applied to the discrete higher-order matrix spectral problem, from which we establish the Hamiltonian structures for the vector generalization of Volterra type differential–difference equations. The outline of the present letter is as follows. In Section 2, based on the discrete zero-curvature equation, a hierarchy of Volterra type differential–difference equations is derived. Then the Hamiltonian forms of differential–difference equations are established by utilizing the discrete trace identity. In Section 3, infinitely many conservation laws for the first system of nonlinear differential–difference equations in the hierarchy are also obtained.

2. A hierarchy of differential–difference equations

Let f be a lattice function, i.e., a function from $\mathbb{Z}$ to $\mathbb{R}$. The shift operator E, the inverse E^{-1} of E and two difference operators Δ, Δ^* are defined as

$$Ef(n) = f(n+1), \qquad E^{-1}f(n) = f(n-1), \qquad \Delta f(n) = (E-1)f(n), \qquad \Delta^* f(n) = (E^{-1}-1)f(n).$$

As normal, we write $f(n) = f,\ f(n\pm 1) = f^{\pm},\ f(n+k) = E^k f,\ n,k \in \mathbb{Z}$. Let us now introduce the following discrete $(N+1) \times (N+1)$ matrix spectral problem

$$E\boldsymbol{\psi} = U\boldsymbol{\psi}, \quad U = \begin{pmatrix} 1+\lambda \boldsymbol{u}^T\boldsymbol{v} & \boldsymbol{u}^T \\ \lambda \boldsymbol{v} & I \end{pmatrix}, \tag{2.1}$$

where $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_{N+1})^T$, λ is a constant spectral parameter, I is a $N \times N$ identity matrix, $\boldsymbol{u} = (u_1, \ldots, u_N)^T$ and $\boldsymbol{v} = (v_1, \ldots, v_N)^T$ are two vector potentials, and $\boldsymbol{u}^T\boldsymbol{v} = u_1v_1 + \cdots + u_Nv_N$. In order to construct the associated differential–difference equations, we first solve the stationary discrete zero-curvature equation:

$$(EV)U - UV = 0, \quad V = \begin{pmatrix} \lambda a & B \\ \lambda C & \lambda A \end{pmatrix}, \tag{2.2}$$

where A denotes $N \times N$ matrix, B denotes $1 \times N$ matrix, C denotes $N \times 1$ matrix, $a = -\mathrm{tr}A$, tr means trace of a matrix. Then (2.2) is equivalent to

$$\begin{aligned} &\Delta B - \lambda(\boldsymbol{u}^T\boldsymbol{v}B + \boldsymbol{u}^T A - a^+\boldsymbol{u}^T) = 0, \\ &\Delta C - \lambda(-\boldsymbol{u}^T\boldsymbol{v}C^+ + a\boldsymbol{v} - A^+\boldsymbol{v}) = 0, \\ &C^+\boldsymbol{u}^T - \boldsymbol{v}B + \Delta A = 0. \end{aligned} \tag{2.3}$$

Expanding a, A, B, C into Laurent polynomials in λ:

$$a = \sum_{j\geqslant 0} a_j\lambda^{-j}, \qquad A = \sum_{j\geqslant 0} A_j\lambda^{-j}, \qquad B = \sum_{j\geqslant 0} B_j\lambda^{-j}, \qquad C = \sum_{j\geqslant 0} C_j\lambda^{-j}, \tag{2.4}$$