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On a result by Dennis and Schnabel for Newton's method: Further improvements

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We improve local convergence results for Newton's method by defining a more precise domain where the Newton iterates lie than in earlier studies using Dennis and Schnabel-type techniques. A numerical example is presented to show that the new convergence radii are larger and new error bounds are more precise than the earlier ones.

1. Introduction

Problems from applied sciences can be written as an equation using Mathematical Modelling of the form

$$F(x) = 0, \tag{1.1}$$

where F is an operator defined on a convex subset Ω of a Banach space B_1 with values in a Banach space B_2.

The most popular iterative method for solving equations is undoubtedly Newton's method

$$x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n), \quad \text{for each } n = 0, 1, 2, \ldots, \tag{1.2}$$

where x_0 is an initial point. We refer the reader to [1–7] and the references there in for local as well as semilocal results for Newton's method. In particular an elegant local result was given by Dennis and Schnabel in [8], for Newton's method. The sufficient convergence conditions are:

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(H_1) Let x^* be a solution of Eq. (1.1) such that the operator $[F'(x^*)]^{-1}$ exists, $B(x^*, r) \subset \Omega$ and $\|[F'(x^*)]^{-1}\| \leq \gamma$, with $r, \gamma > 0$,

(H_2) $\|F''(x)\| \leq M$ for $x \in \Omega$.

Dennis and Schnabel proved under (H_1) and (H_2), that for any starting point in $B(x^*, \varepsilon)$, where $\varepsilon = \min\{r, R\}$ and $R = \frac{1}{2\gamma M}$, Newton's method is convergent. The local results provide what we call ball of convergence, $B(x^*, \varepsilon)$. From the value ε, this ball of convergence gives information about the accessibility of the solution x^* of the equation to solve by the iterative method considered to approximate x^*. In [9], they presented a generalization of (H_2) that consists of considering the condition $\|F''(x)\| \leq \omega(\|x\|)$, $x \in \Omega$, where $\omega : [0, +\infty) \to \mathbb{R}$ is a non-decreasing continuous function such that $\omega(0) \geq 0$. In this paper, we present a generalization of the previous condition but in affine invariant form to high order derivatives of the operator F; in particular, we suppose that

$$\|[F'(x^*)]^{-1}(F'(x) - F'(x^*))\| \leq \omega_0(\|x - x^*\|), \quad x \in \Omega. \tag{1.3}$$

Let $R^* = \sup\{t \in [0, +\infty) : w_0(t) < 1\}$. Moreover, suppose that for each $x \in U(x^*, R^*) \cap \Omega$

$$\|[F'(x^*)]^{-1} F^{(k)}(x)\| \leq \omega(\|x\|), \quad k \geq 3, \tag{1.4}$$

where $\omega, \omega_0 : [0, +\infty) \to \mathbb{R}$ are non-decreasing continuous functions such that $\omega(0) \geq 0$ and $\omega_0(0) \geq 0$. The advantages of presenting the results in affine instead of non-affine invariant form are well known (see, e.g. [2]). As already noted in [9,10] an interesting situation is given when (1.1) is a polynomial equation of degree k, since the operator $F^{(k)}(x)$ is such that $\|[F'(x^*)]^{-1}F^{(k)}(x)\| \leq M$, $x \in \Omega$, and consequently $[F'(x^*)]^{-1}F^{(k)}(x)$ always satisfies conditions (1.4) and (1.3). Even, for more general equations, by using Taylor's series, Eq. (1.1) can be approximated by polynomial equations.

In [10], Argyros and González used condition (2.5) to improve the results in [8,9]. In the present study we improve the earlier results even further. In particular, we show that the new convergence radii are larger than the earlier ones [8–10]. Moreover, the new error bounds are also more precise. It is worth noticing that the improvements are made under the same computational cost as before [8–10].

The paper is organized as follows: in Section 2, we prove a new local convergence result for Newton's method. In Section 3, we present an example to show the advantages of the new approach.

2. Local convergence and order of convergence

We obtain a new local convergence result for Newton's method when the operator F satisfies conditions (1.3) and (1.4). For this, we follow a similar idea to that given in [10].

Theorem 2.1. *Let $F : \Omega \subseteq B_1 \to B_2$ be a nonlinear k ($k \geq 3$) times continuously differentiable operator on a non-empty open convex domain Ω of a Banach space B_1 with values in a Banach space B_2. Let x^* be a solution of $F(x) = 0$ such that the operator $[F'(x^*)]^{-1}$ exists, $B(x^*, r) \subseteq \Omega$ and $\|F'(x^*)^{-1}F^{(i)}(x^*)\| \leq \alpha_i$ (for $i = 2, 3, \ldots, k-1$) with $r, \alpha_i > 0$. Suppose that conditions (1.3) and (1.4) are satisfied and there exists the smallest positive zero R_0 of the equation*

$$\frac{k-1}{k}\left(\sum_{i=1}^{k-2} \frac{\alpha_{i+1}}{i!} t^{i-1} + \frac{t^{k-2}}{(k-1)!}\omega(\|x^*\| + t)\right) t + \omega_0(t) - 1 = 0. \tag{2.1}$$

Then, there exists $\varepsilon > 0$ such that Newton's sequence $\{x_n\}$ is well-defined and converges to x^* for every $x_0 \in B(x^*, \varepsilon)$. Moreover, the following error bounds hold

$$\|x^* - x_n\| \leq \beta \|x^* - x_{n-1}\| \tag{2.2}$$