Applied Mathematics Letters 55 (2016) 54–62

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

# On the optimal parameters of GMSSOR method for saddle point problems

Yanhui Bi[a,*], Naimin Zhang[b], Lijuan Zhou[b]

[a] Information and Engineering College, Dalian University, Dalian, 116622, China
[b] School of Mathematics and Information Science, Wenzhou University, Wenzhou, 325035, China

ARTICLE INFO

*Article history:*
Received 1 September 2015
Received in revised form 24 November 2015
Accepted 24 November 2015
Available online 11 December 2015

*Keywords:*
Saddle point problems
GMSSOR method
Optimal parameters
Convergence

ABSTRACT

For solving saddle point problems, SOR-type methods are investigated by many researchers in the literature. In this short note, we study the GMSSOR method for solving saddle point problems and obtain the optimal parameters which minimize the spectral (or pseudo-spectral) radii of the iteration matrices.



## 1. Introduction

Consider the following augmented linear system:

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ with $rank(B) \le n$, $b \in \mathbb{R}^m$, $q \in \mathbb{R}^n$, and $m \ge n$. We denote the range and the null space of $A$ by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, the transpose of $A$ by $A^T$, respectively.

Linear systems of the form (1.1) are called saddle point problems. They arise in many scientific and engineering applications (cf. [1]), including computational fluid dynamics, constrained optimization, incompressible elasticity, circuit analysis, structural analysis, and so forth. Many iterative techniques, including stationary iterative methods and Krylov subspace iterative methods have been proposed for solving saddle

* Corresponding author.
*E-mail addresses:* biyanhui_dl@sina.com (Y. Bi), nmzhang@wzu.edu.cn (N. Zhang).

http://dx.doi.org/10.1016/j.aml.2015.11.007
0893-9659/© 2016 Published by Elsevier Ltd.

point linear systems in the literature. In this paper we are concerned with SOR-type methods and discuss their optimal parameters for solving saddle point problems.

Golub et al. [2] studied the convergence of successive overrelaxation-like (SORL) methods and discussed the optimal parameters. Bai et al. [3] proposed the generalized successive overrelaxation (GSOR) method. They established the convergence theory for the GSOR method and obtained the optimal parameters when (1.1) is a nonsingular linear system. When $rank(B) < n$, then (1.1) is a singular linear system. Zheng et al. [4] applied the GSOR method to solve singular saddle point problems and discussed the semi-convergence, they also obtained the optimal parameters which made the fastest semi-convergence. Zhang and Lu [5] proposed the generalized symmetric successive overrelaxation (GSSOR) method to solve saddle point problems. Chao et al. [6] obtained the optimal parameters of the GSSOR method for solving nonsingular saddle point problems. Recently, Darvishi et al. [7] and Zhang et al. [8] studied the generalized modified symmetric successive overrelaxation (GMSSOR) method for solving nonsingular saddle point problems, and Zhou et al. [9] discussed the semi-convergence of the GMSSOR method for solving singular saddle point problems. In this note, we further study the optimal parameters of the GMSSOR method for solving singular and nonsingular saddle point problems.

## 2. Preliminary observation for the GMSSOR method

Given initial guess $x^{(0)} \in \mathbb{R}^m$ and $y^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots$, the GMSSOR method [8] computes the approximate solutions of (1.1) iteratively by

$$\begin{cases} y^{(k+1)} = y^{(k)} + \dfrac{4\tau}{2-\tau}Q^{-1}B^T\left[(1-\omega)x^{(k)} - \omega A^{-1}By^{(k)} + \omega A^{-1}b\right] - \dfrac{4\tau}{2-\tau}Q^{-1}q, \\ x^{(k+1)} = (1-\omega)^2x^{(k)} - \omega A^{-1}B\left[y^{(k+1)} + (1-\omega)y^{(k)}\right] + \omega(2-\omega)A^{-1}b, \end{cases} \tag{2.1}$$

where $\omega$ and $\tau$ are two relaxation parameters, and $Q$ is an $n \times n$ symmetric positive definite matrix, which generally acts as an approximate (preconditioning) matrix of the Schur complement $B^TA^{-1}B$.

Let $z^{(k)} = ({x^{(k)}}^T, {y^{(k)}}^T)^T$. Then it is easy to see that the GMSSOR method can also be described as

$$z^{(k+1)} = H(\omega,\tau)z^{(k)} + g, \tag{2.2}$$

and the iteration matrix $H(\omega,\tau)$ is given by

$$H(\omega,\tau) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where

$$\begin{aligned} H_{11} &= (1-\omega)^2I_m - \frac{4\omega\tau(1-\omega)}{2-\tau}A^{-1}BQ^{-1}B^T, \\ H_{12} &= -\omega(2-\omega)A^{-1}B + \frac{4\omega^2\tau}{2-\tau}A^{-1}BQ^{-1}B^TA^{-1}B, \\ H_{21} &= \frac{4\tau(1-\omega)}{2-\tau}Q^{-1}B^T, \qquad H_{22} = I_n - \frac{4\omega\tau}{2-\tau}Q^{-1}B^TA^{-1}B, \end{aligned}$$

and

$$g = \begin{pmatrix} \omega(2-\omega)A^{-1}b - \dfrac{4\omega^2\tau}{2-\tau}A^{-1}BQ^{-1}B^TA^{-1}b + \dfrac{4\omega\tau}{2-\tau}A^{-1}BQ^{-1}q \\ \dfrac{4\omega\tau}{2-\tau}Q^{-1}B^TA^{-1}b - \dfrac{4\tau}{2-\tau}Q^{-1}q \end{pmatrix}.$$