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# Nonlocal symmetries, exact solutions and conservation laws of the coupled Hirota equations

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ABSTRACT

Using the Lax pair, nonlocal symmetries of the coupled Hirota equations are obtained. By introducing an appropriate auxiliary dependent variable, the nonlocal symmetries are successfully localized to Lie point symmetries. With the help of Lie symmetries of the closed prolongation, exact solutions and nonlocal conservation laws of the coupled Hirota equations are studied.



## 1. Introduction

The Lie symmetries [1–3] and their various generalizations have become an important subject in mathematics and physics. One can reduce the dimensions of partial differential equations (PDEs) and proceed to construct analytical solutions by using classical or non-classical Lie symmetries. However, with all its importance and power, the traditional Lie approach does not provide all the answers to mounting challenges of the modern nonlinear physics. In the 80s of the last century, there exist so-called nonlocal symmetries which entered the literature largely through the work of Olver [4]. Compared with the local symmetries, little importance is attached to the existence and applications of the nonlocal ones. The reason lies in that nonlocal symmetries are difficult to find and similarity reductions cannot be directly calculated. Researchers [5–7] have done a lot of work in this area, Refs. [8–12] give a direct way to solve this problem which so-called localization method of nonlocal symmetries. I.e. the original system is prolonged to a larger system such that the nonlocal symmetry of the original model becomes a local one of the prolonged system. When we get the Lie symmetries of prolonged system, they will correspond to a family of group-invariant solutions. These symmetry group techniques [13–15] provide one method for obtaining exact and special solutions of a given PDE in terms of solutions of lower dimensional equations, in particular, ordinary differential equations.

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Conservation laws are used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness and stability analysis. It can lead to some new integrable systems via reciprocal transformation. The famous Noether's theorem [16] provides a systematic way of determining conservation laws, for Euler–Lagrange differential equations, to each Noether symmetry associated with the Lagrangian there corresponds a conservation law which can be determined explicitly by a formula. But this theorem relies on the availability of classical Lagrangians. To find conservation laws of differential equations without classical Lagrangians, researchers have made various generalizations of Noether's theorem. Steudel [17] writes a conservation law in characteristic form, where the characteristics are the multipliers of the differential equations. In order to determine a conservation law one has to also find the related characteristics. Anco and Bluman [18] provides formulae for finding conservation laws for known characteristics. Infinitely many nonlocal conservation laws for$(1+1)$-dimensional evolution equations are revealed by Lou [19]. Symmetry considerations for PDEs were incorporated by Ibragimov [20] which can be computed by a formula.

This paper is arranged as follows: In Section 2, the nonlocal symmetries of the coupled Hirota equations are obtained by using the Lax pair. In Section 3, we transform the nonlocal symmetries into Lie point symmetries. Then, the finite symmetry transformations are obtained by solving the initial value problem. In Section 4, exact group-invariant solutions of the coupled Hirota equations are obtained. In Section 5, based on the symmetries of prolonged system, nonlocal conservation laws of the coupled Hirota equations are given out. Finally, some conclusions and discussions are given in Section 6.

## 2. Nonlocal symmetries of the coupled Hirota equations

The well-known Hirota equation [21] reads

$$iu_t + \alpha(u_{xx} + 2|u|^2 u) + i\beta(u_{xxx} + 6|u|^2 u_x) = 0, \tag{1}$$

$\alpha, \beta$ are real constants. Eq. (1) is the third flow of the nonlinear Schrödinger (NLS) hierarchy which can be used to describe many kinds of nonlinear phenomenas or mechanisms in the fields of physics, optical fibers, electric communication and other engineering sciences. Eq. (1) reduces to NLS equation when $\alpha = 1, \beta = 0$.

In this section, we shall consider the coupled Hirota equations

$$iu_t + \alpha(u_{xx} - 2u^2 v) + i\beta(u_{xxx} - 6uvu_x) = 0, \qquad iv_t - \alpha(v_{xx} - 2v^2 u) + i\beta(v_{xxx} - 6uvv_x) = 0. \tag{2}$$

Eqs. (2) are reduced to Eq. (1) when $u = -v^*$, and $*$ denotes the complex conjugate. The Lax pair of Eq. (2) has been obtained in [22]

$$\Phi_x = U\Phi, \quad U = \begin{pmatrix} -i\lambda & u \\ v & i\lambda \end{pmatrix} \tag{3}$$

and

$$\Phi_t = V\Phi, \quad V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{4}$$

with

$$\begin{aligned} a &= -4\beta i\lambda^3 - 2\alpha i\lambda^2 - 2\beta iuv\lambda - \alpha iuv + \beta(vu_x - uv_x), \\ b &= 4\beta u\lambda^2 + (2\beta iu_x + 2\alpha u)\lambda + \alpha iu_x - \beta(u_{xx} - 2u^2 v), \\ c &= 4\beta v\lambda^2 - (2\beta iv_x - 2\alpha v)\lambda - \alpha iv_x - \beta(v_{xx} - 2v^2 u) \end{aligned} \tag{5}$$

where $u$ and $v$ are two potentials, the spectral parameter $\lambda$ is an arbitrary complex constant and its eigenfunction is $\Phi = (\phi, \psi)^T$.