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Shocks and rarefactions arise in a two-phase model with logistic growth

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ABSTRACT

Multi-phase or mixture models are often used to describe the dynamics of complex fluids. In this work, we use a general transformation to reduce the two-phase system of one spatial and time variable to a system of a single variable. Then we assess the behavior of solutions for the inviscid two-phase model with logistic growth. The growth rate widely impacts the behavior of the solution, producing either shocks or rarefactions. Increasing growth increases the frequency and spread of these waves, and eliminating growth reduces solutions to continuous traveling waves. This analysis generalizes a class of asymptotic/linear results for gel swelling as well as showing the extraordinary richness in the molding framework.

1. Introduction

Multiphase models arise from conservation laws with each phase averaged over a control volume, where the volume-averaged phases are incompressible. These models account for dynamic contribution from the internal structure of each phase (as in classical fluids) as well as interaction forces via intra-phase friction. Two-phase models have been used in a variety of applications where the focus is on emergent properties and fluid/structure interactions. Similar models have also been used to describe the swelling dynamics of polymeric gels [1–3]. Two-phase models capture a variety of behaviors, so we should expect a multitude of exact solutions to exist. However, few mechanisms have been developed to assess particular solutions of this model. As shown by Ekrut et al. [4], transformations found using symmetry analysis exist for the two-phase system without growth. One such transformation produced a particular solution to a free boundary problem and led to a wide variety of traveling wave solutions, but without growth these models are variations of phase-field-type models.

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2. The two-phase model

The underlying assumption is that any infinitesimal volume can be broken down into volume fractions occupied by each phase. Gels are comprised of a polymeric network (with volume fraction ϕ_1) and a fluid solvent (with volume fraction ϕ_2) with associated velocities u and v. Since the entire volume is occupied by the mixture, $\phi_1 + \phi_2 = 1$. Conservation of mass, which is equivalent to conservation of volume if the phases have the same density (as assumed here), implies,

$$\frac{\partial}{\partial t}(\phi_1) + \frac{\partial}{\partial x}(u\phi_1) = G(\phi_1, \phi_2) \tag{1}$$

$$\frac{\partial}{\partial t}(\phi_2) + \frac{\partial}{\partial x}(v\phi_2) = -G(\phi_1, \phi_2), \tag{2}$$

where G accounts for production of mass within the averaged volume. Examples of growing materials include biofilms, where the network is produced by the bacteria [1]. Standard scaling allows us to neglect any inertial terms, so that conservation of momentum is derived by considering the balance forces acting upon each of the phases,

$$\mu_1 \frac{\partial}{\partial x}\left(\phi_1 \frac{\partial}{\partial x} u\right) = -\phi_1 \frac{\partial}{\partial x} P(\phi_1, \phi_2) + \frac{\partial}{\partial x}\psi(\phi_1) + \xi\phi_1\phi_2(u - v), \tag{3}$$

$$\mu_2 \frac{\partial}{\partial x}\left(\phi_2 \frac{\partial}{\partial x} v\right) = -\phi_2 \frac{\partial}{\partial x} P(\phi_1, \phi_2) - \xi\phi_1\phi_2(u - v). \tag{4}$$

The left-hand-sides denote interphase friction (i.e. viscosity), which is balanced by normal forces (i.e. hydrostatic pressure). Intraphase friction represents momentum transfer between the materials if the phases do not move together ($\xi\phi_1\phi_2(u - v)$). We also assume that the network is chemically active and represent the osmotic pressure that acts on the network as $\frac{\partial}{\partial x}\psi(\phi_1)$. Summing (3)–(4) and with $\phi_2 = 1 - \phi_2$, we find the following system.

$$\frac{\partial}{\partial t}(\phi_1) + \frac{\partial}{\partial x}(u\phi_1) = G(\phi_1, 1 - \phi_1), \tag{5}$$

$$-\frac{\partial}{\partial t}(\phi_1) + \frac{\partial}{\partial x}(v(1 - \phi_1)) = -G(\phi_1, 1 - \phi_1), \tag{6}$$

$$\mu_1(1 - \phi_1)\frac{\partial}{\partial x}\left(\phi_1 \frac{\partial}{\partial x} u\right) - \mu_2\phi_1 \frac{\partial}{\partial x}\left((1 - \phi)\frac{\partial}{\partial x} v\right) = Q \tag{7}$$

where

$$Q = \xi\phi_1(1 - \phi_1)(u - v) + k_2\phi \frac{\partial}{\partial x}\phi_1(1 - \phi)(3\phi_1 - 2\phi_0).$$

3. General reduction

As derived in [4], we find the following transformation to reduce (5)–(7).

$$\begin{aligned} u(t, x) &= \frac{\Gamma(t)}{\alpha} + f\left(x - \frac{1}{\alpha}\int \Gamma(t)dt\right), \\ v(t, x) &= \frac{\Gamma(t)}{\alpha} + g\left(x - \frac{1}{\alpha}\int \Gamma(t)dt\right), \\ \phi_1(t, x) &= m\left(x - \frac{1}{\alpha}\int \Gamma(t)dt\right), \end{aligned} \tag{8}$$