Applied Mathematics Letters 52 (2016) 9–14

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

A study of mixed Hadamard and Riemann–Liouville fractional integro-differential inclusions via endpoint theory

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A R T I C L E I N F O

Article history:
Received 1 July 2015
Received in revised form 2 August 2015
Accepted 2 August 2015
Available online 13 August 2015

Keywords:
Hadamard derivative
Riemann–Liouville fractional integral
Inclusions
Endpoint theory

A B S T R A C T

This paper studies the existence of solutions for a mixed initial value problem of Hadamard and Riemann–Liouville fractional integro-differential inclusions by means of endpoint theory. The main result is well illustrated with the aid of example.

1. Introduction

In this paper, we consider the following mixed initial value problem involving Hadamard derivative and Riemann–Liouville fractional integrals given by

$$\begin{cases} D^{\alpha}\left(x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))\right) \in F(t,x(t)), \quad t\in J:=[1,T], \\ x(1)=0, \end{cases} \tag{1.1}$$

where D^{α} denotes the Hadamard fractional derivative of order α, $0<\alpha\le 1$, I^{ϕ} is the Riemann–Liouville fractional integral of order $\phi>0$, $\phi\in\{\beta_1,\beta_2,\ldots,\beta_m\}$, $F: J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$, ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$), $h_i\in C(J\times\mathbb{R},\mathbb{R})$ with $h_i(1,0)=0, i=1,2,\ldots,m$.

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http://dx.doi.org/10.1016/j.aml.2015.08.002
0893-9659/

Initial and boundary value problems of fractional differential equations and inclusions have been extensively investigated in recent years. Differential inclusions are viewed as generalization of differential equations and inequalities, and are found to be quite useful in the study of optimal control theory and stochastic processes [1]. Differential inclusions also play a significant role in investigating dynamical systems having velocities not uniquely determined by the state of the system, though they depend on it. For details and examples of fractional order differential inclusions, we refer the reader to the works [2–6] and the references cited therein. The interest in the study of fractional calculus owes to its extensive applications in different fields of applied sciences and engineering. Examples include finance system [7], biological system [8], rotor-bearing system [9], electrical circuit [10], oscillators [11], viscoelasticity and diffusion processes [12–15], etc. One of the reasons for the popularity of the subject is the memory (hereditary) property of fractional-order operators that has led to several developments, for instance, sliding mode control of fractional-order chaotic systems [16].

We organize the contents of the paper as follows. Section 2 contains some preliminary concepts related to the proposed study while the main existence result is obtained by applying endpoint theory in Section 3. The paper concludes with an illustrative example.

2. Preliminaries

Let us recall some basic definitions of fractional calculus [17].

Definition 2.1. The Hadamard derivative of fractional order q for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(\log \frac{t}{s} \right)^{n-q-1} \frac{g(s)}{s} ds, \quad n-1 < q < n, \ n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. The Hadamard fractional integral of order q for a function g is defined as

$$\mathcal{I}^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.3. The Riemann–Liouville fractional integral of order $p > 0$ of a continuous function $f : (1, \infty) \to \mathbb{R}$ is defined by

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_1^t (t-s)^{p-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(1, \infty)$.

In relation to problem (1.1), we need the following lemma [17].

Lemma 2.1. *Let $g : J \to \mathbb{R}$ and $h : J \times \mathbb{R} \to \mathbb{R}$ be continuous functions, $0 < \alpha \le 1$, $\beta_i > 0$, $i = 1, \dots, m$. Then the unique solution of the fractional initial value problem*

$$\begin{cases} D^\alpha \left(x(t) - \sum_{i=1}^m I^{\beta_i} h_i(t, x(t)) \right) = g(t), & t \in J, \\ x(1) = 0, \end{cases} \tag{2.1}$$