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Global boundedness of solutions to a quasilinear parabolic–parabolic Keller–Segel system with logistic source ☆

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ABSTRACT

We consider a quasilinear parabolic–parabolic Keller–Segel system with a logistic type source $u_t = \nabla \cdot (\phi(u)\nabla u) - \nabla \cdot (\psi(u)\nabla v) + g(u)$, $v_t = \Delta v - v + u$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, subject to nonnegative initial data and homogeneous Neumann boundary conditions, where ϕ, ψ and g are smooth positive functions satisfying $c_1 s^p \leq \phi(s)$ and $c_1 s^q \leq \psi(s) \leq c_2 s^q$ for $p, q \in \mathbb{R}$ and $s \geq s_0 > 1$, $g(s) \leq as - \mu s^k$ for $s > 0$, with constants $a \geq 0$, $\mu, c_1, c_2 > 0$, and the extended logistic exponent $k > 1$ instead of the ordinary $k = 2$. It is proved that if $q < k - 1$, or $q = k - 1$ with μ properly large that $\mu > \mu_0$ for some $\mu_0 > 0$, then there exists a classical solution which is global in time and bounded. This shows the exact way of the logistic exponent $k > 1$ effecting the behavior of solutions.

1. Introduction

In this paper, we consider a quasilinear parabolic–parabolic Keller–Segel system of chemotaxis model with a source term of logistic type,

$$\begin{cases} u_t = \nabla \cdot (\phi(u)\nabla u) - \nabla \cdot (\psi(u)\nabla v) + g(u), & (x,t) \in \Omega \times (0,T), \\ \tau v_t = \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\tau = 1$, and $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. The smooth nonlinear functions ϕ and ψ satisfy

$$\phi, \psi \in C^2([0,\infty)), \qquad \phi(s) > 0 \quad \text{for } s \geq 0, \tag{1.2}$$

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$$c_1 s^p \le \phi(s), \quad s \ge s_0, \tag{1.3}$$

$$\bar{c}_2 s^q \le \psi(s) \le c_2 s^q, \quad s \ge s_0, \tag{1.4}$$

with $p, q \in \mathbb{R}$, $c_1, c_2, \bar{c}_2 > 0$, and $s_0 > 1$, and g is smooth on $[0, \infty)$ fulfilling

$$g(0) \ge 0, \qquad g(s) \le as - \mu s^k, \quad s > 0 \tag{1.5}$$

with $a \ge 0$, $\mu > 0$ and $k > 1$. The nonnegative initial data

$$u_0 \in C^\beta(\bar{\Omega}), \qquad \beta \in (0,1), \qquad v_0 \in W^{1,r}(\Omega), \quad r > n. \tag{1.6}$$

Eq. (1.1) is an extended version of the well-known Keller–Segel system, proposed by Keller and Segel in 1970. The original Keller–Segel system, that is, $\phi \equiv 1$, $\psi \equiv \chi u$ and $g \equiv 0$ in (1.1), was used to describe the cells (with density u) move towards the concentration gradient of a chemical substance v produced by the cells themselves, where the behavior of solutions depends on the interaction between the two mechanisms of diffusion and aggregation, and so a finite time blow-up of u may occur, under the mass conservation, when the aggregation dominates the system. In past decades, the classical Keller–Segel system has been extensively studied with rich dynamics properties of solutions established, such as the global existence (boundedness) versus the finite time blow-up of solutions [1–4].

An important revision to the classical Keller–Segel system was made by Hillen and Painter[5]. They took a volume-effect (i.e. the positive size of the cells is not ignored) into the system by introducing two positive weight functions ϕ and ψ to represent the diffusivity and chemotactic sensitivity respectively, that is the form of Eq. (1.1) with $g = 0$, which has been widely studied as well [6–8]. For instance, in the corresponding parabolic–elliptic case with $\tau = 0$, $g \equiv 0$ in (1.1) and the second equation replaced by $0 = \Delta v - \frac{1}{|\Omega|}\int_\Omega u + u$, a critical exponent on the interplay of ϕ and ψ was obtained that all solutions are globally bounded if $q - p < \frac{2}{n}$, whereas if $q - p > \frac{2}{n}$ and $q > 0$, the model processes radial finite time blow-up solutions [9]. The value $\frac{2}{n}$ is critical also for the fully parabolic system (1.1) with $g \equiv 0$: there exist globally bounded solutions if $q - p < \frac{2}{n}$ [10], whereas if $q - p > \frac{2}{n}$ with $n \ge 2$, unbounded solutions are possible [8], and even finite time blow-up may occur with $n \ge 3$ and $q \ge 1$ [11].

The influence of sources for the classical Keller–Segel system is significant, since the mass conservation is not true anymore with $g \not\equiv 0$. Obviously, the logistic type source $g(s) = s - s^k$ with $k > 1$ plays as a positive source when $s < 1$, and an absorption (a restriction to growth) if $s > 1$. In general, the growth restriction from the logistic source would benefit the global existence and boundedness of solutions to the Keller–Segel models. The case of $k = 2$ in (1.5) has been well studied [12–16]. For the semilinear chemotaxis system when $\phi = 1$ and $\psi = \chi u$ with $\chi > 0$, Tello and Winkler [14] have proved that if $\mu > \frac{n-2}{n}\chi$, then the solutions of the parabolic–elliptic system are global and bounded, and the same is true for the parabolic–parabolic system, when either $n = 2$ [17], or $n \ge 3$ with μ sufficiently large [18]. Moreover, the quasilinear system with ϕ and ψ satisfying (1.2)–(1.4) has been considered in [12,13,9] as well. For instance, with $\tau = 0$ and $\psi = \chi u$ in (1.1), it is known that if $\mu > \chi(1 - \frac{2}{n(1-p)_+})$, the solutions are global and bounded [13]. The original requirement is $p > 1 - \frac{2}{n}$ to ensure the boundedness of solutions without the logistic source [9]. In particular, it is obtained for the fully parabolic system (1.1) with $k = 2$ that whenever $q < 1$ [12], or $q = 1$ with μ sufficiently large [19], there exists a unique classical solution, global in time and bounded, for any sufficiently smooth initial data.

The present paper deals with the more general logistic source with $k > 1$ instead of $k = 2$. We will show in what way the value of $k > 1$ effects the behavior of solutions to (1.1). The main result is the following theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $n \ge 1$, ϕ, ψ and g satisfy (1.2)–(1.5), with nonnegative initial data (1.6). If $q < k - 1$, or $q = k - 1$ with μ properly large that $\mu > \mu_0$ for some $\mu_0 > 0$, then (1.1) uniquely admits a classical solution (u, v) such that $u \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$*