Applied Mathematics Letters 52 (2016) 46–52

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

# Numerical methods for a quadratic matrix equation with a nonsingular M-matrix[☆]

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ARTICLE INFO

*Article history:*
Received 6 July 2015
Received in revised form 12 August 2015
Accepted 12 August 2015
Available online 21 August 2015

*Keywords:*
Quadratic matrix equation
M-matrix
Nonsymmetric algebraic Riccati equation
Fixed-point iteration

ABSTRACT

We consider numerical solution of a quadratic matrix equation associated with a nonsingular M-matrix (M-QME), which arises in study of noisy Wiener–Hopf problems for Markov chain. We first transform the M-QME to a nonsymmetric algebraic Riccati equation (NARE) of special form, and then solve this special NARE by fixed-point iteration. Theoretical analysis and numerical experiments show that our method is effective and efficient.



## 1. Introduction and preliminaries

In this paper, we consider the quadratic matrix equation (QME) of the form

$$X^2 - EX - F = 0, \tag{1}$$

where $E,\ F \in \mathbb{R}^{n\times n}$, $E = diag(e_1, e_2, \ldots, e_n)$ is a diagonal matrix and $F = (f_{ij})$ is a nonsingular M-matrix. The QME of this special form is called M-QME. The study of M-QME is motivated by noisy Wiener–Hopf problems for Markov chains, where a specific Q-matrix (-Q is an M-matrix) is needed to satisfy M-QME (1), see [1,2] for more details.

[2] showed that M-QME (1) has a unique nonsingular M-matrix solution, which is of practical interest. Some methods have been developed for solving M-QME by transforming it into an equivalent nonsymmetric algebraic Riccati equation (NARE) to solve (see [2–4]). In this paper, our main aim is to develop a more efficient transformation method.

☆ Supported in part by National Natural Science Foundation of China grant 10261012 and 11428104.

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http://dx.doi.org/10.1016/j.aml.2015.08.006

In the following, we first review some basic results on M-matrices and regular splitting.

For any matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times n}$, we write $A \geq B(A > B)$, if $a_{ij} \geq b_{ij}(a_{ij} > b_{ij})$ for all $i, j$. $A$ is called a Z-matrix if $a_{ij} \leq 0$ for all $i \neq j$. A Z-matrix $A$ is called an M-matrix if there exists a nonnegative matrix B such that $A = sI - B$ and $s \geq \rho(B)$. Here and in the following, $\rho(C)$ denotes the spectral radius of $C$. In particular $A$ is called a nonsingular M-matrix if $s > \rho(B)$ and singular M-matrix if $s = \rho(B)$. The following results on M-matrices can be found in [5,6].

**Lemma 1.1.** *Let $A$ be a Z-matrix, then the following statements are equivalent:*

(1) *$A$ is a nonsingular M-matrix;*

(2) *There exists a vector $v > 0$ such that $Av > 0$.*

**Lemma 1.2.** *If $A$ is an irreducible nonsingular M-matrix, then $A^{-1} > 0$.*

Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, then $A = M - N$ is called a splitting of $A$ if $M$ is nonsingular; a regular splitting if $M$ is nonsingular, $M^{-1} \geq 0$, and $N \geq 0$. The following lemma is a basic result for regular splitting (see [6]).

**Lemma 1.3.** *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of $A$, where $A$ is nonsingular and $A^{-1} \geq 0$. If $N_2 \geq N_1 \geq 0$, then $1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) \geq 0$. If, moreover, $A^{-1} > 0$ and if $N_2 \geq N_1 \geq 0$, equality excluded, then $1 > \rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 0$.*

We now review some basic results on nonsymmetric algebraic Riccati equation (NARE)

$$XCX - XD - AX + B = 0, \tag{2}$$

where $A$, $B$, $C$ and $D$ are real matrices of sizes $m \times m$, $m \times n$, $n \times m$ and $n \times n$ respectively. The NARE of this kind appears in transport theory, Wiener–Hopf factorization of Markov chains and etc., for which one can refer to [7,8,2,9] and the references therein. For the NARE (2), the solution of practical interest is its (entrywise) minimal nonnegative solution.

**Lemma 1.4** (*[8]*). *From the NARE (2), we define an $(m + n) \times (m + n)$ matrix*

$$M = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}. \tag{3}$$

*If $M$ is a nonsingular M-matrix, then (2) has a unique minimal nonnegative solution $S$, and both $D - CS$ and $A - CS$ are nonsingular M-matrices. Moreover, both $D - CS$ and $A - CS$ are irreducible when $M$ is an irreducible (singular or nonsingular) M-matrix.*

There have been many effective methods proposed for solving numerically NARE (2) with $M$ being an M-matrix, see [7,3,8,10]. Among them, the fixed-point iteration is the simplest and feasible one.

To compute the M-matrix solution of M-QME (1), Guo in [2] turned it into a special NARE to solve by the transformation

$$X = \alpha I - Y. \tag{4}$$

The motivational thought behind this is that many effective numerical methods for NARE can be fully utilized for M-QME (1). In this paper, we use a more general transformation than (4) to expect to achieve better effectiveness.

The rest of the paper is organized as follows. In Section 2, a new transformation is introduced to turn M-QME (1) into a special NARE and theoretical analysis is given. In Section 3, the special NARE is solved by