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On the non-existence of higher order monotone approximation schemes for HJB equations

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ABSTRACT

In this work we present a result on the non-existence of monotone, consistent linear discrete approximation of order higher than 2. This is an essential ingredient, if we want to solve numerically nonlinear and particularly Hamilton–Jacobi–Bellman (HJB) equations.

1. Introduction

The Hamilton–Jacobi–Bellman (HJB) equation, as well as other nonlinear PDEs may not have solution in the classical sense. Therefore, Crandall, Ishii and Lions [1] introduced in 1992 the concept of viscosity solution, suitable for HJB equations. For a brief introduction to the theory of viscosity solutions we refer to [2]. However, it can be a problem to find even such solution analytically, therefore numerical method schemes are used [3–5].

The classical theory proposed by Barles and Souganidis [6], which is widely used for proving the convergence of numerical schemes for HJB equations, is based on the monotonicity of the underlying scheme. Convergent schemes for problems from mathematical finance are often first order-accurate in time, and first or second-order accurate in space. Recently, Wang and Forsyth [4] proposed an approach yielding an accuracy close to second order in space. In this work, we prove that no better result can exist.

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2. Definitions

Let us first introduce the basic notations. U denotes a suitable function space. Let $FV(x)$, $F : U \to \mathbb{R}$ be any nonlinear differential operator, and

$$GV(x) = G\big(V(x), V(x + b_1 h), V(x + b_2 h), \ldots, V(x + b_n h)\big) \tag{1}$$

be the corresponding discrete scheme approximating it. $V(x)$ is defined as a possibly multidimensional function with suitable properties, b_i, $i = 1, 2, \ldots, n$ is of the same dimension as x, and the uniform step size $h \in \mathbb{R}^+$.

Definition 1 (*Monotonicity*)**.** A discrete approximation scheme (1) is monotone, if the function G is non-increasing in $V(x + b_i h)$ for $b_i \neq 0$, $i = 1, \ldots, n$.

Definition 2 (*Standard Consistency*)**.** The discrete scheme

$$GV(x) = G\big(V(x), V(x + b_1 h), V(x + b_2 h), \ldots, V(x + b_n h)\big)$$

is a consistent approximation of $FV(x)$, if $\lim_{h\to 0} \|FV(x) - GV(x)\|_\infty = 0$, where $V(x)$ is a solution of the equation $FV(x) = 0$. Further, $GV(x)$ is said to be consistent of order $p > 0$, if $\|FV(x) - GV(x)\|_\infty = \mathcal{O}(h^p)$, $h \to 0$.

However, the equation $FV(x) = 0$ may not possess classical solutions, which turns Definition 2 inapplicable. For example, HJB equations often have solutions only in the viscosity sense. Therefore, we use another definition of consistency that does not use a solution of $FV(x) = 0$:

Definition 3 (*Consistency in Viscosity-sense*)**.** The discrete scheme

$$G\phi(x) = G(\phi(x), \phi(x + b_1 h), \phi(x + b_2 h), \ldots, \phi(x + b_n h))$$

is a consistent approximation of $FV(x)$ if $\lim_{h\to 0} \|F\phi(x) - G\phi(x)\|_\infty = 0$, for any smooth test function $\phi(x)$. We say it is consistent of order $p > 0$, if $\lim_{h\to 0} \|F\phi(x) - G\phi(x)\|_\infty = \mathcal{O}(h^p)$ for any smooth test function $\phi(x)$.

Let $V(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a locally C^2-function ($x, y \in \mathbb{R}$ are now one-dimensional). We define the differential operator $\mathcal{L} : C^2(\mathbb{R}^2) \to C(\mathbb{R}^2)$

$$\mathcal{L}V(x, y) = \alpha_1 \frac{\partial^2 V}{\partial x^2} + \alpha_{12} \frac{\partial^2 V}{\partial x \partial y} + \alpha_2 \frac{\partial^2 V}{\partial y^2} + \beta_1 \frac{\partial V}{\partial x} + \beta_2 \frac{\partial V}{\partial y} + \gamma V. \tag{2}$$

We assume $\alpha_1 \neq 0$ and investigate some properties of the linear operator $L : C^2(\mathbb{R}^2) \to C(\mathbb{R}^2)$ given by

$$\begin{aligned} LV(x, y) &= a_0(h)V(x, y) + a_1(h)V(x + b_1 h, y + c_1 h) \\ &\quad + a_2(h)V(x + b_2 h, y + c_2 h) + \cdots + a_n(h)V(x + b_n h, y + c_n h), \end{aligned} \tag{3}$$

where $b_i \neq 0$, or $c_i \neq 0$, $i = 1, 2, \ldots, n$ and there exist j, k such that $b_j \neq 0$, $c_k \neq 0$. Eq. (3) should be an approximation of the differential operator $\mathcal{L}V(x, y)$.

Definition 4 (*Positive coefficients approximation*)**.** The linear discrete approximation scheme (3) satisfies the positive coefficients condition if $a_i(h) \geq 0$ for $i = 1, 2, \ldots, n$, for all $h > 0$.

Often a scheme is monotone, if and only if its linear part satisfies positive coefficient condition.