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Relationships between different types of initial conditions for simultaneous root finding methods

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ABSTRACT

The construction of initial conditions of an iterative method is one of the most important problems in solving nonlinear equations. In this paper, we obtain relationships between different types of initial conditions that guarantee the convergence of iterative methods for simultaneously finding all zeros of a polynomial. In particular, we show that any local convergence theorem for a simultaneous method can be converted into a convergence theorem with computationally verifiable initial conditions which is of practical importance. Thus, we propose a new approach for obtaining semilocal convergence results for simultaneous methods via local convergence results.

1. Introduction and preliminaries

Throughout this paper $(\mathbb{K}, |\cdot|)$ denotes an algebraically closed normed field, $\mathbb{K}[z]$ denotes the ring of polynomials over $\mathbb{K}$, and the vector space $\mathbb{K}^n$ is equipped with the p-norm $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for some $1 \le p \le \infty$.

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \ge 2$. We consider the zeros of f as a vector in $\mathbb{K}^n$. More precisely, a vector $\xi \in \mathbb{K}^n$ is said to be a *root-vector* of f if $f(z) = a_0 \prod_{i=1}^n (z - \xi_i)$ for all $z \in \mathbb{K}$, where $a_0 \in \mathbb{K}$. Without doubt the most famous iterative method for simultaneously finding all the zeros of a polynomial f is the Weierstrass method, which is defined by

$$x^{k+1} = x^k - W_f(x^k), \quad k = 0, 1, 2, \ldots, \tag{1.1}$$

where the Weierstrass correction $W_f \colon \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is defined by

$$W_f(x) = (W_1(x), \ldots, W_n(x)) \quad \text{with } W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \ne i} (x_i - x_j)} \ (i = 1, \ldots, n), \tag{1.2}$$

where a_0 is the leading coefficient of f and $\mathcal{D}$ is the set of all vectors in $\mathbb{K}^n$ with distinct components.

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Let us consider three classical convergence theorems for the Weierstrass method. In these theorems, we assume that f is a complex polynomial of degree $n \geq 2$ which has only simple zeros, and that $\xi \in \mathbb{C}^n$ is a root-vector of f. Throughout the paper we use the function $\delta\colon \mathbb{K}^n \to \mathbb{R}_+$ defined by $\delta(x) = \min_{i \neq j} |x_i - x_j|$ and the function $d\colon \mathbb{K}^n \to \mathbb{R}^n$ defined by

$$d(x) = (d_1(x), \ldots, d_n(x)), \quad \text{where } d_i(x) = \min_{j \neq i} |x_i - x_j|. \tag{1.3}$$

Theorem A (*Dochev [1]*). *If $x^0 \in \mathbb{C}^n$ is an initial guess such that*

$$\|x^0 - \xi\|_\infty < \frac{\sqrt[n-1]{2} - 1}{2\sqrt[n-1]{2} - 1}\, \delta(\xi), \tag{1.4}$$

then the Weierstrass iteration (1.1) converges quadratically to ξ.

Theorem B (*Wang and Zhao [2]*). *If $x^0 \in \mathbb{C}^n$ is an initial guess such that*

$$\|x^0 - \xi\|_\infty < \frac{\sqrt[n-1]{2} - 1}{4\sqrt[n-1]{2} - 3}\, \delta(x^0), \tag{1.5}$$

then the Weierstrass iteration (1.1) converges to ξ.

Theorem C (*Petković, Carstensen and Trajković [3]*). *If $x^0 \in \mathbb{C}^n$ is an initial guess with distinct components such that*

$$\|W_f(x^0)\|_\infty < \frac{\delta(x^0)}{5n}, \tag{1.6}$$

then the Weierstrass iteration (1.1) converges to a root-vector of f.

Both sides of the initial condition of Theorem A depend on the desired root-vector ξ which is unknown. The initial condition of Theorem B also contains unknown data, but only on the left-hand side of (1.5). Usually, we say that these results are rather of theoretical importance. The initial condition of Theorem C is of significant practical importance since it depends only on available data: the coefficients of f, the degree n and the initial guess x^0.

Surprisingly, each of these theorems is a consequence of the previous one. It turns out that this situation is not accidental. Among the other results, we prove that from any theorem of type A, we can obtain a theorem of type B as well as a theorem of type C. Besides, from any theorem of type B we can obtain a theorem of type C.

The main purpose of this paper is to show that any local convergence theorem for a simultaneous method can be converted into a convergence theorem with computationally verified conditions. In other words, in this area both local and semilocal convergence results are of significant practical importance. Our results are based on a new localization theorem for polynomial zeros.

2. Initial conditions for convergence of simultaneous methods

For given vectors $x \in \mathbb{K}^n$ and $y \in \mathbb{R}^n$, we define in $\mathbb{R}^n$ the vector

$$\frac{x}{y} = \left(\frac{|x_1|}{y_1}, \ldots, \frac{|x_n|}{y_n} \right),$$

provided that y has no zero components. Given p such that $1 \leq p \leq \infty$, we denote by q the conjugate exponent of p, i.e. q is defined by means of

$$1 \leq q \leq \infty \quad \text{and} \quad 1/p + 1/q = 1.$$