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A note on terminal value problems for fractional differential equations on infinite interval

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ABSTRACT

In this note we establish sufficient conditions for existence and uniqueness of solutions of terminal value problems for a class of fractional differential equations on infinite interval. Some illustrative examples are comprehended to demonstrate the proficiency and utility of our results.

1. Introduction

Fractional differential equations have been of awesome enthusiasm for as far back as three decades and have discovered various applications in different fields of physical science and engineering [1]. We can find various applications in viscoelasticity, electrochemistry, control, porous media and electromagnetics. Fractional calculus and fractional differential equations have encountered augmented study starting late as a noteworthy interest both in mathematics and in applications.

The existence theory for initial value problems including fractional differential equation has obtained extensive consideration amid late decades [2–13]. The existence hypothesis for boundary value problems of fractional differential equations on infinite interval has been considered extensively, see [14–24].

Terminal value problems for differential equation now a days play an essential part in the modeling of numerous phenomena in physical science, engineering, and so forth. Terminal value problems emerge naturally in the simulation of techniques that are watched (i.e. measured) at a later point, eventually after the methodology has started. Existence theory for classical terminal value problems has been investigated by

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several researchers [25–27]. The related theory of terminal value problems for fractional differential equations is more complicated and has received attention quite recently. In particular K. Diethelm [28,29] established existence uniqueness and stability results for fractional terminal value problems on finite interval and it has been observed that initial value problems and terminal value problems have in general different solution structure. In many cases the domain of governing equation for certain physical phenomena is an infinite or semi-infinite interval. Infinite or semi-infinite interval problems require special treatment. The existence theory for infinite interval problems involving fractional derivative has been studied by number of authors [30–33]. However, to the best of our knowledge the existence hypothesis for terminal value problems for fractional differential equation on infinite interval is never been studied so far. Motivated by [26], the objective of this work is to establish conditions for the existence and uniqueness of the solutions for terminal value problem for fractional differential equations of the type

$$\begin{aligned} D^{\alpha}_{*\infty^-}u(x) &= f(x,u(x),u'(x)), \quad 1<\alpha\le 2,\ x\in[a,\infty) \\ u(\infty) &= \mu, \qquad u'(\infty)=0, \end{aligned} \tag{1.1}$$

where non-linear function $f:[a,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is assumed to be continuous and u' is derivative of u. The rest of paper is organized as follows. In Section 2, we shall present some necessary definitions and preliminary results that will be used throughout the paper. In Section 3, we shall prove some lemmas and existence theorem and apply the Banach fixed point theorem to establish a uniqueness result. Also few examples are included to show the applicability of our result.

2. Preliminaries

In this section, we introduce some definitions, notations and give several lemmas, which will be used in this paper (see [34,35]).

Definition 2.1. Let $\alpha>0$ and $f\in C[a,\infty)$. The operator $I^{\alpha}_{b-}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b(s-x)^{\alpha-1}f(s)\,ds$ is called right Riemann–Liouville fractional integral of order $\alpha>0$. Furthermore the operator $I^{\alpha}_{\infty-}f(x):=\lim_{b\to\infty}I^{\alpha}_{-b}f(x)$ is called right-Liouville fractional integral.

Definition 2.2. Let $\alpha>0$ and $n=\lceil\alpha\rceil$. Assume $f\in C^n[a,\infty)$, then right Caputo fractional derivative of f of order α is defined as $D^{\alpha}_{*b-}f(x)=(-1)^n I^{n-\alpha}_{b-}\frac{d^n}{dx^n}f(x)$ on $[a,\infty)$ and $D^{\alpha}_{*\infty-}f(x):=\lim_{b\to\infty}D^{\alpha}_{*b-}f(x)$ on $[a,\infty)$.

Lemma 2.3. *Assume that $1<\alpha\le 2$ and $f\in C^2[a,\infty)$ then $I^{\alpha}_{b-}D^{\alpha}_{*b-}f(x)=f(x)-[f(b)+f'(b)(b-x)]$.*

Lemma 2.4. *If f is continuous and as $1<\alpha\le 2$, then $D^{\alpha}_{*\infty-}I^{\alpha}_{\infty-}f=f$.*

Let $C^1[0,\infty)$ be the space of continuously differentiable functions on $[0,\infty)$ and norm $\|.\|$ defined as $\|u\|=\sup_{x\in[a,\infty)}|u(x)|+\sup_{x\in[a,\infty)}|u'(x)|$ and $\mathbb{X}=\{u(x):u\in C^1[a,\infty)\}$. Then $(\mathbb{X},\|.\|)$ is a Banach space. Following assumptions will be used in the sequel to establish existence results for solutions of terminal value problem of fractional differential equation.

- (H_1) $f\in C(P);\ f(x,u_1,u_1')\ge f(x,u_2,u_2')$ for $u_1\ge u_2,\ u_1'\le u_2'$, where $P:=\{(x,u,u'):x\in[a,\infty),\lim_{x\to\infty}u(x)$ and $\lim_{x\to\infty}u'(x)$ exist$\}$ and $C(P)$ is a set of continuous functions defined on P,
- (H_2) $f\in C(P);\ f(x,0,0)=0$ and $\lim_{x\to\infty}\int_x^{\infty}(s-x)^{\alpha-2}f(s,u(s),u'(s))\,ds=0$ for $x\in[a,\infty)$,
- (H_3) $|f(x,u,u')-f(x,v,v')|\le L(x)[|u-v|+|u'-v'|]$, where $L(x)$ is measurable function on $[a,\infty)$.