Applied Mathematics Letters 52 (2016) 132–141

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

On Jacobi's condition for the simplest problem of calculus of variations with mixed boundary conditions

Milan Batista

University of Ljubljana, Faculty of Maritime Studies and Transport, Slovenia

ARTICLE INFO

Article history:
Received 24 June 2015
Received in revised form 18 August 2015
Accepted 18 August 2015
Available online 5 September 2015

Keywords:
Calculus of variations
Jacobi's condition
Mixed boundary condition

ABSTRACT

The purpose of this paper is an extension of Jacobi's criteria for positive definiteness of second variation of the simplest problems of calculus of variations subject to mixed boundary conditions. Both non constrained and isoperimetric problems are discussed. The main result is that if we stipulate conditions (21) and (22) then Jacobi's condition remains valid also for the mixed boundary conditions.
© 2015 Elsevier Ltd. All rights reserved.

1. Introduction

As it is well known [1–7] the simplest problem of the calculus of variations is to find a function $y = y(x)$ that minimize (or maximize) the functional

$$J[y] \equiv \int_a^b F(x, y, y')\, dx \tag{1}$$

under Dirichlet boundary conditions

$$y(a) = A \text{ (fixed)}, \qquad y(b) = B \text{ (fixed)} \tag{2}$$

where function F and constants A and $B$are given, $()' \equiv d()/dx$. For convenience we shall call this problem the Dirichlet problem.

Note. In the paper we shall assume that for all the functions we are going to use the domain of definition is the interval $[a, b]$ and that they possess continuous derivatives with respect to all its arguments as many order as needed, unless stated otherwise. We introduce a function by writing $y = y(x)$(as example) and in the sequel use y to denote the function and $y(x)$ to denote its value.

E-mail address: milan.batista@fpp.uni-lj.si.

http://dx.doi.org/10.1016/j.aml.2015.08.010
0893-9659/© 2015 Elsevier Ltd. All rights reserved.

Suppose that y is an extremal of $J[y]$. Then a sufficient condition for y to realize a weak minimum of $J[y]$ is that its second variation $\delta^2 J[h]$ is positive definite, that is, $\delta^2 J[h] > 0$ for any piecewise smooth, i.e., continuous and piecewise continuously differentiable, $h = h(x) \not\equiv 0$ such that

$$h(a) = 0, \qquad h(b) = 0. \tag{3}$$

In this paper we will assume that $\delta^2 J[h]$ is given in the integrated form

$$\delta^2 J[h] \equiv Rh^2\Big|_a^b + \int_a^b \left(Ph'^2 + Qh^2\right) dx \tag{4}$$

where

$$P(x) \equiv \frac{\partial^2 F}{\partial y'^2}, \qquad Q(x) \equiv \frac{\partial^2 F}{\partial y^2} - \frac{dR}{dx}, \qquad R(x) \equiv \frac{\partial^2 F}{\partial y \partial y'}. \tag{5}$$

Here and in the sequel the partial derivatives are evaluated at (x, y, y'). We abbreviate the condition $\delta^2 J[h] > 0$ as $\delta^2 J > 0$ to mean that it must be fulfilled for piecewise smooth function $h = h(x) \not\equiv 0$ which satisfies the constraints of the given variation problem. We will call such h an admissible variation.

Now the question under what conditions we have $\delta^2 J > 0$, the answer is the following Jacobi's theorem (see for instance [1,4–6]).

Theorem 1 (*Dirichlet Problem*). *In order that $\delta^2 J > 0$ it is necessary and sufficient that the following conditions hold:*

1. *$P(x) > 0$ for all $a \le x \le b$ (strengthened Legendre condition)*
2. *$u(x) \neq 0$ for all $a < x \le b$ (strengthened Jacobi condition)*

where $u = u(x)$ is the solution of Jacobi's accessory equation

$$\mathbf{L}(u) \equiv -\frac{d}{dx}(Pu') + Qu = 0 \tag{6}$$

which satisfies the initial conditions

$$u(a) = 0, \qquad u'(a) = 1. \tag{7}$$

Note that $u'(a) = 1$ is only for sake of definiteness of u [5].

For a simplest isoperimetric problem y must beside boundary conditions (2) satisfy also the additional condition

$$\int_a^b G(x, y, y')\, dx = \ell \tag{8}$$

where ℓ is a given constant. In this case an extremal of the problem y is obtained from the functional

$$J[y] = \int_a^b H(x, y, y')\, dx \tag{9}$$

where $H \equiv F + \lambda G$, λ is a Lagrange multiplier. The requirement for $\delta^2 J > 0$ which is given by (4) and where (5) is replaced by

$$P(x) \equiv \frac{\partial^2 H}{\partial y'^2}, \qquad Q(x) \equiv \frac{\partial^2 H}{\partial y^2} - \frac{dR}{dx}, \qquad R(x) \equiv \frac{\partial^2 H}{\partial y \partial y'} \tag{10}$$

Download English Version:

https://daneshyari.com/en/article/1707565

Download Persian Version:

https://daneshyari.com/article/1707565

Daneshyari.com