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Existence of positive periodic solutions of first order neutral differential equations with variable coefficients

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ABSTRACT

This work deals with the existence of positive ω-periodic solutions for the first order neutral differential equation. The results are established using Krasnoselskii's fixed point theorem. An example is given to support the theory.

1. Introduction

In the present work, we give new sufficient conditions for the existence of positive ω-periodic solutions of the following first-order neutral differential equation

$$[x(t) - P(t)x(t-\tau)]' = -Q(t)x(t) + f(t, x(t-\tau)), \tag{1}$$

where $Q \in C(\mathbb{R}, (0,\infty))$, $P \in C^1(\mathbb{R},\mathbb{R})$, $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\tau > 0$, and P, Q are ω-periodic functions, f is ω-periodic with respect to first variable.

In recent years, there has been considerable interest in the existence of positive periodic solutions of first order neutral differential equations. These equations appear in the blood cell production models, population models and control models. In [1], existence of positive periodic solutions of the following neutral differential equation

$$\frac{d}{dt}[x(t) - cx(t-\tau(t))] = -a(t)x(t) + f(t, x(t-\tau(t))) \tag{2}$$

were investigated when $0 \leqslant c < 1$ and $-1 < c < 0$. In the present paper, we have two main contributions comparing with the existing results. First, instead of constant c we take variable coefficient $P(t)$. Second, in

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addition to $0 \leqslant P(t) < 1$ and $-1 < P(t) \leqslant 0$, we consider the ranges $1 < P(t) < \infty$ and $-\infty < P(t) < -1$ for $P(t)$, which are new in the literature. There are also some other studies dealing with positive solutions of neutral differential equations, see, [2–6] and references therein. For related books, we refer the reader to [7–11].

The following fixed point theorem will be used in proofs.

Lemma 1 (*Krasnoselskii's Fixed Point Theorem [10]*). *Let X be a Banach space, let Ω be a bounded closed and convex subset of X and, let S_1, S_2 be maps of Ω into X such that $S_1x+S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contractive and S_2 is completely continuous, then the equation*

$$S_1x + S_2x = x$$

has a solution in Ω.

2. Main results

Let $\Phi = \{x(t) : x(t) \in C(\mathbb{R}, \mathbb{R}),\ x(t) = x(t + \omega),\ t \in \mathbb{R}\}$ with the sup norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. It is clear that Φ is a Banach space.

Theorem 1. *Assume that $1 < p_0 \leqslant P(t) \leqslant p_1 < \infty$ and that there exist constants m and M such that*

$$(p_1 - 1)m \leqslant P(t)x - \frac{f(t, x)}{Q(t)} \leqslant (p_0 - 1)M, \quad \forall (t, x) \in [0, \omega] \times [m, M],\ m > 0. \tag{3}$$

Then (1) has at least one positive ω-periodic solution $x(t) \in [m, M]$.

Proof. It is well known that to find an ω-periodic solution of (1) is equivalent to find an ω-periodic solution of the integral equation

$$x(t) = \frac{1}{P(t+\tau)}\left[x(t+\tau) + \int_{t+\tau}^{t+\tau+\omega} G(t+\tau, s)\left[P(s)Q(s)x(s-\tau) - f(s, x(s-\tau))\right] ds\right],$$

where

$$G(t, s) = \frac{\exp(\int_t^s Q(u)du)}{\exp(\int_0^\omega Q(u)du) - 1}.$$

Let $\Omega = \{x \in \Phi : m \leqslant x(t) \leqslant M, t \in [0, \omega], 0 < m < M\}$. One can observe that Ω is a bounded, closed and convex subset of Φ. We define two mappings $S_1, S_2 : \Omega \to \Phi$ as follows

$$(S_1x)(t) = \frac{1}{P(t+\tau)}\int_{t+\tau}^{t+\tau+\omega} G(t+\tau, s)\left[P(s)Q(s)x(s-\tau) - f(s, x(s-\tau))\right] ds, \tag{4}$$

$$(S_2x)(t) = \frac{x(t+\tau)}{P(t+\tau)}. \tag{5}$$

For any $x \in \Omega$ and $t \in \mathbb{R}$, we have from (4) and (5) that

$$\begin{aligned}(S_1x)(t+\omega) &= \frac{1}{P(t+\tau+\omega)}\int_{t+\tau+\omega}^{t+\tau+2\omega} G(t+\tau+\omega, s)\Big[P(s)Q(s)x(s-\tau) - f(s, x(s-\tau))\Big] ds \\ &= \frac{1}{P(t+\tau+\omega)}\int_{t+\tau}^{t+\tau+\omega} G(t+\tau+\omega, u+\omega)\Big[P(u+\omega)Q(u+\omega)x(u+\omega-\tau) \\ &\quad - f(u+\omega, x(u+\omega-\tau))\Big] du\end{aligned}$$