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Subharmonic solutions with prescribed minimal period of an impulsive forced pendulum equation[☆]

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ABSTRACT

This paper is concerned with a second-order impulsive forced pendulum equation. With the least action principle and some techniques of mathematical analysis, we show that the equation has at least one periodic solution with the specified minimal period.

1. Introduction

We consider the classical forced pendulum equation with impulsive effects,

$$\begin{cases} \ddot{x}(t) + A\sin x(t) = f(t), \quad t \neq t_j,\ t \in \mathbb{R}, \\ \Delta\dot{x}(t_j) = I_j(x(t_j)), \quad j \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $A = g/l$ is a constant with g being the gravity constant and l being the length of the pendulum, $f : \mathbb{R} \to \mathbb{R}$ is a T-periodic function which is regarded as an external force, $I_j \in C(\mathbb{R},\ \mathbb{R})$ for each $j \in \mathbb{Z}$, and the operator Δ is defined as $\Delta\dot{x}(t_j) = \dot{x}(t_j^+) - \dot{x}(t_j^-)$, where $\dot{x}(t_j^+)$ ($\dot{x}(t_j^-)$) denotes the right-hand (left-hand) limit of $\dot{x}$ at t_j. There exists an $m \in \mathbb{N}$ such that $0 = t_0 < t_1 < \cdots < t_m = T$, $t_{j+m} = t_j + T$, and $I_{j+m} = I_j$, $j \in \mathbb{Z}$.

A function $x(t) \in C(\mathbb{R}, \mathbb{R})$ is a solution of system (1.1) if the function $x(t)$ satisfies (1.1). For any integer $p \geq 2$, a solution $x(t)$ to (1.1) is called a pth-order subharmonic solution if $x(t)$ is periodic with the minimal period pT in t.

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When $I_j = 0$, Eq. (1.1) reduces to the following equation.

$$\ddot{x}(t) + A \sin x(t) = f(t), \quad t \in \mathbb{R}. \tag{1.2}$$

Since Hamel studied it in 1922, many researchers were interested in Eq. (1.2). See [1–3] and the references therein. Particularly, Yu [4] recently obtained some appropriated results about it by using a new decomposition technique.

When $I_j = d_j,\ d_j \in \mathbb{R}$, Chen, Li and He [5] studied the existence of subharmonic solutions about (1.1).

However, to the best of our knowledge, few researchers discussed the existence of subharmonic solutions with prescribed minimal period for forced pendulum equations with impulsive condition being not constants. This motivates us to consider the impulsive forced pendulum equation (1.1).

In the literature, some classical tools such as fixed point theorems in cones, topological degree theory, the upper and lower solutions method combined with monotone iterative technique [6–10] have been widely used to get solutions of impulsive differential equations. The variational approach to the study of impulsive differential equations possibly is due to Nieto and O'Regan (2007) [11], who constructed a variational structure and converted the problem on the existence of solutions of a second-order impulsive equation to that on the existence of critical points of the corresponding variational functional. Since then, many important results have been obtained, such as boundary value problems [12–14], periodic solutions [15,16], homoclinic solutions [17].

Variational methods are employed in this paper and we get subharmonic solutions as critical points of a certain functional defined in a Hilbert space.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, under suitable hypotheses, we show that Eq. (1.1) possesses at least one periodic solution with minimal period pT by using the least action principle. In Section 4, with the help of an example, we demonstrate the applicability of our main results.

2. Preliminaries

In the following, we first introduce some notations. Define the space

$$X = \left\{x : \mathbb{R} \to \mathbb{R} | x,\ \dot{x} \in L^2([0,T],\mathbb{R}), x(t) = x(t+pT),\ t \in \mathbb{R},\ \text{where } p \text{ is an integer and } p \geq 2\right\}.$$

Then X is a Hilbert space and the inner product

$$(x,y) = \int_0^{pT} (\dot{x}(t)\dot{y}(t) + x(t)y(t))dt, \quad x, y, \in X,$$

induces the norm

$$\|x\|_X = \left(\int_0^{pT} (x'(t)^2 + x^2(t))dt\right)^{\frac{1}{2}}.$$

Next we set $\Omega = \{1, 2, \ldots, pm-1\}$ and define a functional φ on X by

$$\varphi(x) = \frac{1}{2}\int_0^{pT} |\dot{x}(t)|^2 dt - A\int_0^{pT} (1-\cos x(t))dt + \int_0^{pT} f(t)x(t)dt + \sum_{j\in\Omega}\int_0^{x(t_j)} I_j(s)ds \tag{2.1}$$

for $x \in X$. Note that φ is Fréchet differentiable at any $x \in X$ and

$$\varphi'(x)(y) = \int_0^{pT} (\dot{x}(t)\dot{y}(t) - Ay(t)\sin x(t) + f(t)y(t))dt + \sum_{j\in\Omega} I_j(x(t_j))y(t_j), \tag{2.2}$$

for $y \in X$. It is clear that critical points of the functional φ are classical pT-periodic solutions of Eq. (1.1).