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# Blow-up phenomena for some nonlinear pseudo-parabolic equations

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ABSTRACT

This paper considers the blow-up of solutions for equations

$$u_t - \nu\triangle u_t = \mathrm{div}(\rho(|\nabla u|^2)\nabla u) + f(u)$$

by means of a differential inequality technique. A lower bound for blow-up time is determined if blow-up does occur. Also, we establish a blow-up criterion and an upper bound for blow-up under some conditions. Moreover, conditions which ensure that blow-up cannot occur are presented. This result extends the results obtained by R. Xu (2007) and P. Luo (2015).



## 1. Introduction

In this paper, we consider the initial–boundary value problem

$$\begin{aligned} &u_t - \nu\triangle u_t - \mathrm{div}(\rho(|\nabla u|^2)\nabla u) = f(u) \quad \text{in } \Omega \times (0,T), \\ &u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x) \geqslant 0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^n (n \geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$.

It is known that for certain classes of functions $\rho$ and $f$, the solution of (1.1) can fail to exist globally only if it blows up at some finite time $T$. Obviously, if $\nu = 0, \rho = 1$, then Eq. (1.1) reduces to the heat equation with sources

$$u_t - \triangle u = f(u). \tag{1.2}$$

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For Eq. (1.2), many results for blow-up of solutions have been obtained [1,2]. In [3] and [4], Payne and Schaefer used a differential inequality technique to obtain a lower bound on blow-up time for Eq. (1.2) under homogeneous Dirichlet boundary conditions and homogeneous Neumann conditions, respectively. When nonlinear conditions are imposed on the boundary, lower bounds were also obtained in [5] for Eq. (1.2). In [6], lower bounds for the blow-up time in a semilinear parabolic problem with a variable exponential source were derived.

If $\nu = 0$, then Eq. (1.1) reduces to be the nonlinear parabolic equation

$$u_t = \operatorname{div}(\rho(|\nabla u|^2)\nabla u) + f(u). \tag{1.3}$$

Applying the differential inequality technique, the authors in [7] derived a lower bound on blow-up time when blow-up does occur, exhibited criteria which imply that blow-up cannot occur, and presented a sufficient condition for blow-up to occur. For nonlinear boundary condition case, the reader can refer to [8] for details. For a more general equation with time dependent coefficients which can be transformed into the form of (1.3), one can see [9].

If $\nu = 1, \rho = 1$, then Eq. (1.1) reduces to be the pseudo-parabolic equation

$$u_t - \triangle u - \triangle u_t = f(u), \tag{1.4}$$

which have been extensively investigated [10–15]. Especially, Xu [14] proved that there are solutions that blow up in finite time $T$ in $H_0^1(\Omega)$-norm and Luo [15] obtained a lower bound in $H_0^1(\Omega)$-norm.

As far as we know, there is little information on the bounds for blow up time to problem (1.1). Inspired by [8,15], here we consider the initial–boundary problem (1.1) and give some results for this problem. A lower bound for blow up time is determined if the solution blows up in finite time. Also, we establish a blow-up criterion and an upper bound for blow-up time under some conditions as well as a nonblow-up. Our results extend the recent results obtained by R. Xu [12] and P. Luo [15]. In detail, this paper is organized as follows: in Section 2, we give nonblow-up case; in Section 3, we determine a lower bound for blow up time to the initial–boundary problem (1.1) under some conditions; in Section 4, we obtain a sufficient condition which guarantees that blow-up occurs at some finite time $T$ and determine an upper bound for $T$.

## 2. Nonblow-up case

We assume that $\rho$ is a positive $C^1$ function which satisfies

$$\rho(s) + 2s\rho'(s) \geqslant 0, \quad s > 0, \tag{2.1}$$

so that $\operatorname{div}(\rho(|\nabla u|^2)\nabla u)$ is elliptic. We claim that $\rho$ and $f$ satisfy the conditions

$$0 < f(s) \leqslant a_1 + a_2 s^p, \qquad \rho(s) \geqslant b_1 + b_2 s^q, \quad s > 0, \tag{2.2}$$

where $1 < p \leqslant \frac{n+2}{n-2}$ and $a_1, a_2, b_1, b_2$ are positive constants (although $a_1 = 0$ is allowable). In addition, we assume that $u_0$ satisfies the compatibility condition $u_0(x) = 0$ on $\partial\Omega$. Since the initial data $u_0(x)$ is nonnegative, we have by the maximum principles [16] that the solution is nonnegative in its time interval of existence. Hereafter, for simplicity, we set $\nu = 1$.

We start with the following local existence theorem for Eq. (1.1) which can be obtained by Faedo–Galerkin methods.

**Theorem 2.1.** *Assume (2.1) and (2.2) hold. Then for any $u_0 \in H_0^1(\Omega)$, there exists a $T > 0$ for which problem (1.1) has a unique local solution*

$$u \in L^\infty([0,T), H_0^1(\Omega)), \qquad u_t \in L^2([0,T), H_0^1(\Omega)),$$