Applied Mathematics Letters 56 (2016) 29–33

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

On the maximum and antimaximum principles for the beam equation

P. Drábek, G. Holubová*

Department of Mathematics and NTIS, University of West Bohemia, Technická 8, 306 14 Plzeň, Czech Republic

ARTICLE INFO

Article history:
Received 2 November 2015
Received in revised form 8 December 2015
Accepted 9 December 2015
Available online 17 December 2015

Keywords:
Fourth order problem
Beam operator
Positive solutions
Negative solutions
Convergent iterations

ABSTRACT

We consider a one-dimensional beam equation with a strictly positive forcing term. Under different assumptions on the restoring force we prove the existence of positive and negative solutions, respectively. Our method is based on non-monotone but convergent iterations and extends related results known in the literature so far.

1. Introduction and statement of the results

Positive and/or negative solutions of the problem

$$\begin{aligned} &u^{(4)} + c(x)u = h(x) \quad \text{in } (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{1}$$

with $c = c(x)$, $h = h(x)$ being continuous functions on $[0,1]$ and $h(x) \geq 0$, $x \in [0,1]$, play the important role in a qualitative analysis of many mathematical models. For all, we can mention models of suspension bridges introduced by Lazer and McKenna [1] and studied later by many authors (see, e.g., [2] and references therein). In particular, a nonlinear one-dimensional problem

$$\begin{aligned} &u^{(4)} + c(x)u^{+} = h(x) \quad \text{in } (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{2}$$

* Corresponding author.
E-mail addresses: pdrabek@kma.zcu.cz (P. Drábek), gabriela@kma.zcu.cz (G. Holubová).

http://dx.doi.org/10.1016/j.aml.2015.12.009
0893-9659/

describes the stationary behavior of the roadbed of the suspension bridge and the positivity of its solution is crucial information also for possible non-stationary solutions of a more complicated time-dependent model. Here, u denotes the deflection of the roadbed, u^+ is the positive part of u, $h(x)$ represents the external loading (of constant sign), and the coefficient $c(x)$ can be understood as a variable stiffness of the bridge ropes (cable stays). In this context, only values $c(x) \geq 0$ are physically relevant, however, from the mathematical point of view, we do not lay any restrictions on the sign of $c(x)$ in further text. Notice that both problems (1) and (2) coincide for u nonnegative.

In order to make our statements precise, we have to introduce the property of strict inverse positivity and negativity. Namely, consider the set

$$W := \{u \in C^4([0,1]) : \; u(0) = u(1) = u''(0) = u''(1) = 0\},$$

and the linear operator $L_c : \; W \to C([0,1])$ defined by

$$L_c u = u^{(4)} + c(x)u, \quad u \in W.$$

Then (1) is equivalent to $L_c u = h$, and, we say that L_c is *strictly inverse positive* (SIP) on W if $u \in W$, $L_c u = h \gneqq 0$ in $[0,1]$ implies $u > 0$ in $(0,1)$, $u'(0) > 0$, $u'(1) < 0$. The definition of a *strict inverse negative* (SIN) operator is similar. Let us note that the SIP and SIN properties are also known as the *maximum* and *antimaximum principles* (especially in connection with the second order problems).

For a continuous function $f = f(x)$ on $[0,1]$ let us denote

$$f_m := \min_{x \in [0,1]} f(x), \qquad f^m := \max_{x \in [0,1]} f(x).$$

As usual, we will also write $f(x) = f^+(x) - f^-(x)$, where $f^\pm(x) = \max\{\pm f(x), 0\}$.

Set $c_0 := 4k_0^4$ with k_0 being the smallest positive solution of the equation $\tan k = \tanh k$ (i.e, $k_0 \approx 3.9266$ and $c_0 \approx 950.8843$). Let us recall the following results.

Proposition 1 (*Schröder [3]*)**.** *Let $-\pi^4 < c(x) \leq c_0$. Then L_c is* SIP *on W. Moreover, if $c(x) \equiv c$ (constant), then L_c is* SIP *on W if and only if $-\pi^4 < c \leq c_0$.*

Proposition 2 (*Cabada, Cid and Sanchez [4]*)**.** *Let $-\frac{c_0}{4} \leq c(x) < -\pi^4$. Then L_c is* SIN *on W. Moreover, if $c(x) \equiv c$ (constant), then L_c is* SIN *on W if and only if $-\frac{c_0}{4} \leq c < -\pi^4$.*

Let us note that the value π^4 is nothing else but the first eigenvalue of the operator L_0 with the corresponding eigenfunction being strictly positive on $(0,1)$. On the other hand, the value c_0 is the maximal value for which the Green function corresponding to L_c with constant c does not change sign. Detailed explanation and other related results concerning also higher dimensional cases can be found, e.g., in [5,6] and in references therein.

In our previous paper [7] we have proved that for non-constant coefficient $c = c(x)$ the inequalities $c(x) > -\pi^4$ in Proposition 1 and $c(x) < -\pi^4$ in Proposition 2 are necessary ones neither for SIP nor for SIN of L_c on W. In particular, in [7, Th. 4] we show that there exists $c = c(x)$ with c_m arbitrarily large negative such that L_c is SIP on W and similarly, in [7, Th. 5] we show that there exists $c = c(x)$ with c^m arbitrarily large positive such that L_c is SIN on W.

In real problems the external load is often strictly positive rather than nonnegative. The following result illustrates that in a special case with a constant positive load h and a constant coefficient c, the existence of a positive solution of (1) requires even no limitation on c from above.

Proposition 3 (*McKenna and Walter [8]*)**.** *Let $h(x) \equiv h > 0$ and $c(x) \equiv c > -\pi^4$. Then problem (1) has a unique positive solution.*