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Oscillation of third-order nonlinear neutral differential equations

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ABSTRACT

We establish some new criteria for the oscillation and asymptotic behaviour of solutions of the third-order nonlinear neutral differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\left[x(t)+a(t)x\big(\gamma(t)\big)\right]'\right)'\right)'+q(t)f\big(x\big(\delta(t)\big)\big)=0.$$

Applications illustrating the results are given.

1. Introduction

Consider third-order neutral differential equations of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\left[x(t)+a(t)x\big(\gamma(t)\big)\right]'\right)'\right)'+q(t)f\big(x\big(\delta(t)\big)\big)=0, \tag{E}$$

where $t \ge t_0$. We will always assume that

(i) $p(t)$, $r(t)$, $q(t)$, $a(t)$, $\gamma(t)$, $\delta(t) \in C[t_0,\infty)$, $p(t)$, $r(t)$, $q(t)$, $\gamma(t)$, $\delta(t)$ are positive for $t \ge t_0$,
(ii) $\int_{t_0}^{\infty} p(t)\,\mathrm{d}t = \int_{t_0}^{\infty} r(t)\,\mathrm{d}t = \infty$,
(iii) $\gamma(t) \le t$, $\lim_{t\to\infty}\gamma(t) = \infty$,
(iv) $\lim_{t\to\infty}\delta(t) = \infty$
(v) $0 \le a(t) \le a_0 < 1$ for $t \ge t_0$,
(vi) $f \in C(\mathbb{R},\mathbb{R})$, f is odd, $f(v)v > 0$ for $v \ne 0$.

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In recent years, a great deal of work has been done on various aspects of differential equations of third order. These equations arise in the study of entry-flow phenomenon, a problem of hydrodynamics, in mathematical theory of thyroid–pituitary interaction, gravity driven flows and three-layer beams. Moreover, Eq. (E) can be viewed as a system of three differential equations of the first order. This system can be applied to models concerning networks containing lossless or shunted transmission lines. Additionally, one can transform system of partial differential equations to an ordinary neutral differential equation, see e.g. monographs [1] and [2].

The aim of this paper is to establish new oscillation criteria for Eq. (E). Recently, a considerable attention was devoted to this topic by many authors, we refer the reader to [3–7] and references given there.

It will be convenient to set for each solution x of (E)

$$u(t) = x(t) + a(t)x\big(\gamma(t)\big). \tag{1}$$

If u is a function defined by (1), then functions

$$u^{[0]} = u, \qquad u^{[1]} = \frac{1}{r(t)}u', \qquad u^{[2]} = \frac{1}{p(t)}\left(\frac{1}{r(t)}u'\right)' = \frac{1}{p(t)}\big(u^{[1]}\big)'$$

are called quasiderivatives of u. A solution x of (E) is said to be *proper* if it exists on the interval $[t_0, \infty)$ and satisfies the condition $\sup\{|x(s)|: t \le s < \infty\} > 0$ for any $t \ge t_0$. A proper solution is called *oscillatory* or *nonoscillatory* according to whether it does or does not have arbitrarily large zeros.

By a modification of the well-known result of Kiguradze we know that every nonoscillatory solution of (E) belongs to one of the following classes of solutions

$$\mathcal{N}_0 = \Big\{x \text{ solution}, \exists T_x\colon u(t)u^{[1]}(t) < 0,\ u(t)u^{[2]}(t) > 0 \text{ for } t \ge T_x\Big\},$$
$$\mathcal{N}_2 = \Big\{x \text{ solution}, \exists T_x\colon u(t)u^{[1]}(t) > 0,\ u(t)u^{[2]}(t) > 0 \text{ for } t \ge T_x\Big\},$$

where u is defined by (1).

The basic properties of solutions in the class $\mathcal{N}_2$ are described by the following lemma.

Lemma 1. *Assume that x is a solution of (E) from the class $\mathcal{N}_2$. Then*

$$(1 - a_0)|u(t)| \le |x(t)| \le |u(t)| \tag{2}$$

for $t \ge T$ and

$$\lim_{t\to\infty} |u(t)| = \lim_{t\to\infty} |x(t)| = \infty. \tag{3}$$

Proof. Let $x \in \mathcal{N}_2$. Without loss of generality we may assume that x is eventually positive, i.e. there exists $T \ge t_0$ such that $x(t) > 0$, $u(t) > 0$, $u^{[1]}(t) > 0$ and $u^{[2]}(t) > 0$ for $t \ge T$. Since $\gamma(t) \le t$ and u is an increasing function, we have $x\big(\gamma(t)\big) \le u\big(\gamma(t)\big) \le u(t)$. Hence

$$x(t) = u(t) - a(t)x\big(\gamma(t)\big) \ge u(t) - a_0 x\big(\gamma(t)\big) \ge u(t) - a_0 u\big(\gamma(t)\big) \ge u(t)(1 - a_0).$$

To prove a second part, we note that as $u^{[1]}$ is positive and increasing function there exists $K > 0$ such that $u^{[1]}(t) \ge K$ for large t. Integrating this inequality from T to t we get

$$u(t) \ge u(T) + K\int_T^t r(s)\,\mathrm{d}s.$$

Letting $t \to \infty$ and using the fact that $\int_{t_0}^{\infty} r(t)\,\mathrm{d}t = \infty$, we obtain $u(t) \to \infty$. By (2), $x(t) \ge (1 - a_0)u(t)$. From this it follows that $x(t) \to \infty$. □