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Positive solutions for a fractional boundary value problem

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ABSTRACT

We obtain a new upper estimate for the Green's function associated with a higher order fractional boundary value problem. As an application of this result, criteria for the existence of positive solutions of the problem are then established.

1. Introduction

In 2010, Goodrich [1] studied the boundary value problem (BVP, for short) consisting of the equation

$$-D^{\nu}_{0+}u(t) = g(t,u), \quad t \in (0,1), \tag{1.1}$$

and the boundary conditions

$$u^{(j)}(0) = 0, \quad j = 0,\dots,n-2, \qquad [D^{\alpha}_{0+}u(t)]_{t=1} = 0, \tag{1.2}$$

where $n \geq 3$, $n-1 < \nu \leq n$, $1 \leq \alpha \leq n-2$, $g : [0,1] \times [0,\infty) \to \mathbb{R}$ is a continuous function, and D^{β}_{0+} is the Riemann–Liouville fractional derivative of order β, i.e.,

$$D^{\beta}_{0+}y(t) = \frac{1}{\Gamma(k-\beta)}\frac{d^k}{dt^k}\int_0^t \frac{y(s)}{(t-s)^{\beta+1-k}}ds, \quad k = [\beta]+1.$$

By a *positive solution* of BVP (1.1), (1.2), we mean a function $u : [0,1] \to [0,\infty)$ such that $u(t)$ satisfies (1.1) and (1.2) and $u(t) > 0$ for $t \in (0,1]$. In [1], the author first obtained some properties of the Green's function associated with the problem. Then, applying these properties and the well known Krasnosel'skii fixed point

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theorem in cones, he derived sufficient conditions for the existence of at least one positive solution of the problem.

Boundary value problems for fractional differential equations have attracted a great deal of attention in the last ten years. As a small sampling of recent results, we refer the reader to the papers of Agarwal, O'Regan, and Stanek [2], Ahmad and Sivasundaram [3], Bai and Lü [4], El-Shahed and Nieto [5], and Wei, Dong, and Che [6]. Basic properties of the fractional calculus and it applications can be found in Kilbas, Srivastava, and Trujillo [7] and Podlubny [8].

As presented in [1], the Green's function for our problem is given by

$$G(t,s)=\frac{1}{\Gamma(\nu)}\begin{cases}t^{\nu-1}(1-s)^{\nu-\alpha-1}-(t-s)^{\nu-1}, & 0\le s\le t\le 1,\\ t^{\nu-1}(1-s)^{\nu-\alpha-1}, & 0\le t\le s\le 1.\end{cases} \tag{1.3}$$

We observe that

$$G(1,s)=\frac{1}{\Gamma(\nu)}((1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}).$$

Lemmas 1.1 and 1.2 give some important properties of $G(t,s)$. Parts (a)–(c) of Lemma 1.1 are taken from [1, Theorem 3.2] and part (d) of Lemma 1.1 is proved in [9, Lemma 3.2]. Lemma 1.2 is taken from [1, Theorem 3.1].

Lemma 1.1. *$G(t,s)$ has the following properties:*

(a) *$G(t,s)$ is a continuous function on $[0,1]\times[0,1]$;*
(b) *$G(t,s)\ge 0$ for $t,s\in[0,1]$;*
(c) *$\max_{t\in[0,1]} G(t,s)=G(1,s)$ for $s\in[0,1]$;*
(d) *$t^{\nu-1}G(1,s)\le G(t,s)\le \frac{1}{\Gamma(\nu)}t^{\nu-1}(1-s)^{\nu-\alpha-1}$ for $t,s\in[0,1]$.*

Lemma 1.2. *Let $l\in C[0,1]$. Then $u(t)$ is a solution of the BVP consisting of the equation*

$$-D^{\nu}_{0+}u(t)=l(t),\quad t\in(0,1), \tag{1.4}$$

and BC (1.2) if and only if

$$u(t)=\int_0^1 G(t,s)l(s)ds. \tag{1.5}$$

In the next section, we will obtain some new sharper upper estimates for G than the ones given in [1] or [9]. In Section 3, we apply our new found estimate to obtain criteria for the existence of a positive solution to problem (1.1), (1.2).

2. A new upper estimate for the Green's function

Let

$$b(t):=\frac{t^{\nu-2}((\nu-1)(1-t)+\alpha t)}{\alpha}.$$

The function $b(t)$ will be used to give our new upper estimate for the Green's function (1.3). The following lemmas provide some useful estimates that will be used in what follows.

Lemma 2.1. *We have*

$$b(t)\ge t^{\nu-1}\quad \textit{for } 0\le t\le 1.$$