Applied Mathematics Letters 56 (2016) 72–77

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Breathers and rogue waves for a third order nonlocal partial differential equation by a bilinear transformation

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ARTICLE INFO

Article history:
Received 5 November 2015
Received in revised form 21 December 2015
Accepted 22 December 2015
Available online 6 January 2016

Keywords:
Nonlocal evolution equations
Rogue waves

ABSTRACT

Breathers and rogue waves as exact solutions of a nonlocal partial differential equation of the third order are derived by a bilinear transformation. Breathers denote families of pulsating modes and can occur for both continuous and discrete systems. Rogue waves are localized in both space and time, and are obtained theoretically as a limiting case of breathers with indefinitely large periods. Both entities are demonstrated analytically to exist for special classes of nonlocal equations relevant to optical waveguides.

1. Introduction

Nonlocal equations refer here broadly to classes of partial differential equations where the spatial rates of change of a function at any point, as well as its time evolution, are related to values at a finite distance from the point under investigation. Such equations have been studied frequently in applied disciplines, e.g. spatiotemporal solitary waves [1] and Hermite–Gaussian beams [2] are considered in nonlocal optical media.

Theoretically, many equations in the theory of nonlinear waves and solitons can be studied through the perspective of nonlocal equations. As illustrative examples, a coupled system of Burgers equations can be rewritten as a single component differential–integral Burgers equation with a translational kernel [3]. Similarly, Boussinesq equations with rational nonlinearity can also be formulated as a single component nonlocal form and soliton expressions are deduced [4]. Indeed many cases studied consist of a diffusion or Schrödinger type differential equation combined with an integral operator with a translational kernel, e.g. (A = complex valued envelope),

$$iA_t + A_{xx} + N(I)A = 0, \qquad I = \text{intensity} = |A|^2, \qquad N(I)A = A\int R(x'-x)I(x')dx', \tag{1}$$

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http://dx.doi.org/10.1016/j.aml.2015.12.016

where $R(x)$ is a response function. If $R(x) = \delta(x)$ (a delta function), this nonlinear term will reduce to a local evolution model, the conventional nonlinear Schrödinger equation,

$$iA_t + A_{xx} + |A|^2 A = 0. \tag{2}$$

Eq. (1) and its variants occur in many applications, e.g. in life science phenomena where the emergence and evolution of biological species are critical issues [5].

The focus here is on a class of nonlocal nonlinear Schrödinger equations. The goal in earlier works is to examine collapse prevention and soliton stabilization [6]. Recently the attention tends to be placed on equations with 'parity–time symmetry' [7]. These 'PT-symmetric' systems are important as theoretically the self-induced potential is then invariant. In terms of applications, wave propagation in symmetric waveguides and photonic lattices has been demonstrated experimentally [7]. An external potential can also be incorporated [8]. A general framework for coupled nonlinear Schrödinger equations can be formulated [9]. Finally, this whole idea can be extended to equations with two or more spatial variables [10].

The objective here is to show that both breathers and rogue waves, intensively studied topics recently, can be derived analytically for these nonlocal nonlinear Schrödinger equations. Breathers are pulsating modes and rogue waves are unexpectedly large amplitude displacements from a tranquil background [11]. Rogue waves were first noted in the oceans by sailors and researchers in fluid mechanics, but are now being pursued in optics and other fields as well [12]. The Darboux transformations have been frequently used for computing rogue waves for many models [13], e.g. the Hirota equation, a member from the nonlinear Schrödinger family with third order dispersion [14]. Recently, the bilinear method has also been shown to be applicable as well, e.g. for the derivative nonlinear Schrödinger equation [15].

The structure of this paper can now be explained. The new nonlocal, third order partial differential equation is formulated and the background for the bilinear transformation is reviewed (Section 2). The expansion scheme for a breather is given and the rogue wave mode is derived by taking a long wave limit (Section 3). The analogy with other evolution equations exhibiting rogue wave modes is highlighted and conclusions are drawn (Section 4).

2. A nonlocal third order nonlinear Schrödinger equation

Consider the nonlocal equation

$$iA_t + icA_x + A_{xx} + \sigma A\left[A\left(-x,t\right)\right]^* A + i\lambda A_{xxx} + i\lambda 3\sigma A\left[A(-x,t)\right]^* A_x = 0, \tag{3}$$

where A is a complex valued wave envelope, the parameters λ, σ, and c are real and '$*$' denotes the complex conjugate. If $c = \lambda = 0$, Eq. (3) reduces to a nonlocal equation studied earlier [7]:

$$iA_t + A_{xx} + \sigma A[A(-x,t)]^* A = 0, \tag{4}$$

which possesses the usual appealing features of soliton equations, e.g. a Lax pair and an infinite number of conservation laws. Furthermore, the elegant mechanism of inverse scattering is also applicable to Eq. (4). The conventional (local) nonlinear Schrödinger equation is recovered if $-x$ is replaced by x in Eq. (4).

The conventional nonlinear Schrödinger equation has an 'integrable' higher order extension which incorporates a third order derivative, commonly known as the Hirota equation, after the equation was first proposed by R. Hirota in 1973 [16]:

$$iA_t + A_{xx} + \sigma AA^* A + i\lambda(A_{xxx} + 3\sigma AA^* A_x) = 0. \tag{5}$$

It will be natural to search for the nonlocal extension of Eq. (5) and we propose that Eq. (3) will be the ideal candidate. It will be illuminating to demonstrate that breathers and rogue wave modes also exist for this class of nonlocal evolution equations with peculiar x, t symmetry. Future works will focus on attempts