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On approximate solutions of some delayed fractional differential equations

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ABSTRACT

In this paper we propose an approach to Ulam's stability of some fractional differential equations with a delay. In particular, we show that, under suitable assumptions, every approximate solution of such an equation is close to a unique exact solution to it. We also obtain an auxiliary result on Ulam's stability of a Volterra integral equation.

1. Introduction

The fractional differential equations are useful tools in the modeling of many physical phenomena and processes in economics, chemistry, aerodynamics, etc. (for more details see [1–4]). In many cases it is very difficult to give a satisfactory description of solutions to such an equation and it is much easier to provide a description of approximate solutions. So, there arises a natural question when we can replace an approximate solution by an exact solution to the equation (or conversely) and what error we thus commit. Some convenient tools to study such dependence can be found in the theory of Ulam's (often also called the Hyers–Ulam) type stability. The notion of that stability (for various types of equations) had been motivated by a problem asked by Ulam in 1940 (cf. [5,6]), concerning approximate homomorphisms of groups, and a solution to it that was published in 1941 by Hyers [5]. For more updated information we refer to [7,8]; for examples of recent results on such stability for differential equations and further references see [9–17].

In this paper $\mathbb{F}$ is either the field of reals $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, $C_J(\mathbb{F})$ denotes the family of all continuous functions mapping a real interval J into $\mathbb{F}$, $\alpha \in (0, 1)$, t_0 and $h > 0$ are fixed real numbers,

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I is a real interval of one of the forms: $[t_0, \infty)$, $[t_0, a)$, $[t_0, a]$ (with some $a > t_0$), $I_h := I \cup [t_0 - h, t_0]$, $\phi : [t_0 - h, t_0] \to \mathbb{F}$ and $f : I \times C_{[-h,0]}(\mathbb{F}) \to \mathbb{F}$ are fixed continuous functions ($C_{[-h,0]}(\mathbb{F})$ is endowed with the supremum norm), and given $y \in C_{I_h}(\mathbb{F})$, we define $y_t \in C_{[-h,0]}(\mathbb{F})$ for $t \in I$ by $y_t(\theta) := y(t + \theta)$ for $\theta \in [-h, 0]$.

We mainly focus on Ulam's stability of the delayed fractional differential equation

$$D^{\alpha}[y(t)g(t)] = f(t, y_t), \quad t \in I, \qquad y(t) = \phi(t), \quad t \in [t_0 - h, t_0], \tag{1.1}$$

with $\mathbb{F} = \mathbb{R}$ and a suitable given function g, where D^{α} stands for the Caputo fractional derivative. Let us recall that the Caputo fractional derivative, of order κ, of a suitable function $h : I \to \mathbb{R}$, is given by

$$^{c}D^{\kappa}_{t_0^+} h(t) = \frac{1}{\Gamma(n-\kappa)} \int_{t_0}^{t} (t-s)^{n-\kappa-1} h^{(n)}(s) ds, \qquad t \in I,$$

where $n = [\kappa] + 1$ ($[\kappa]$ denotes the integer part of κ) and Γ is the Gamma function.

A particular case of (1.1) has been investigated in [18], with $g(t) \equiv e^{\beta t}$, f written as $f(t, y_t) \equiv f_0(t, y_t)e^{\beta t}$ with some continuous $f_0 : I \times C_{[-h,0]}(\mathbb{R}) \to \mathbb{R}$ and $\beta > 0$ being fixed real number. In such a case equation (1.1) is equivalent (cf. [19,18]) to the following integral equation

$$\begin{aligned} y(t) &= y(t_0)e^{\beta(t_0 - t)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{\beta(s-t)} f_0(s, y_s) ds, \qquad t \in I, \\ y(t) &= \phi(t), \qquad t \in [t_0 - h, t_0]. \end{aligned} \tag{1.2}$$

A theorem on the existence of solutions to (1.2) has been provided in [18].

We investigate Ulam's type stability of (1.1). We also prove an auxiliary result on such stability of a Volterra type integral equation (2.4), which corresponds to numerous related outcomes in [20–26].

2. Main result

In what follows, $b : I \times \mathbb{F} \to \mathbb{F}$ is continuous with respect to the first variable, $b(t_0, s) = s$ for $s \in \mathbb{F}$, $K \in (0, 1)$, $\psi : I_h \to \mathbb{R}_+$ (positive reals) is continuous, $\psi_h : I \to \mathbb{R}_+$ is given by $\psi_h(t) := \sup_{s \in [t-h,t]} \psi(s)$, $I_2 := \{(t, s) \in I^2 : s < t\}$, $N : I_2 \times C_{[-h,0]}(\mathbb{F}) \to \mathbb{F}$ is continuous, and $L : I_2 \times [0, \infty) \to [0, \infty)$ is nondecreasing with respect to the third variable and continuous. Moreover, we assume that

$$\int_{t_0}^{t} L(t, s, \gamma \psi_h(s)) ds \le K \gamma \psi(t), \qquad t \in I, \gamma \in \mathbb{R}_+, \tag{2.1}$$

$$|N(t, s, z_s) - N(t, s, w_s)| \le L(t, s, \|z_s - w_s\|), \qquad (t, s) \in I_2, z, w \in C_{I_h}(\mathbb{F}). \tag{2.2}$$

The following theorem is an auxiliary result; the proof of it is provided in the next section. It is considered in a more general situation (where $\mathbb{F}$ can be also the set of complex numbers) than it is necessary in the proof of our main result, because of its relations with the outcomes in [20–26].

Theorem 2.1. *Let $y \in C_{I_h}(\mathbb{F})$, $y(t) = \phi(t)$ for $t \in [t_0 - h, t_0]$ and*

$$\left| y(t) - b(t, y(t_0)) - \int_{t_0}^{t} N(t, s, y_s) ds \right| \le \psi(t), \qquad t \in I. \tag{2.3}$$

Then there is a unique $\widehat{y} \in C_{I_h}(\mathbb{F})$ such that $\widehat{y}(t) = \phi(t)$ for $t \in [t_0 - h, t_0]$,

$$\widehat{y}(t) = b(t, \widehat{y}(t_0)) + \int_{t_0}^{t} N(t, s, \widehat{y}_t) ds, \qquad t \in I, \tag{2.4}$$

$$|\, y(t) - \widehat{y}(t) \,| \le \frac{\psi(t)}{1 - K}, \qquad t \in I. \tag{2.5}$$