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On some isoperimetric inequalities involving eigenvalues of fixed membranes

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ABSTRACT

In 1956, Hersch (1965) derived some isoperimetric inequalities for eigenvalues of a fixed membrane Ω, simply connected, with a center of symmetry **O**. In this note we are going to derive some sharper versions of Hersch's results. More precisely, if Ω is symmetric of order 2 we show that we have $\lambda_2(\Omega)+\lambda_3(\Omega) \le 2\lambda_2(D_{r_o})$, where D_{r_o} is a disc of radius $r_o(\Omega)$ (the conformal radius of Ω at **O**). Also, if Ω is symmetric of order 4, we have $\lambda_4(\Omega)+\lambda_5(\Omega) \le 2\lambda_4(D_{r_o})$.

1. Introduction

This note deals with the classical eigenvalue problem for the Dirichlet Laplacian:

$$\begin{cases} \Delta\varphi+\lambda\varphi=0 & \text{in } \Omega, \\ \varphi=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded simply connected domain with boundary $\partial\Omega$. It is well known that this problem has non-trivial solutions only for specific values of λ called eigenvalues, that may be arranged in an increasing sequence $0<\lambda_1(\Omega)<\lambda_2(\Omega)\le\lambda_3(\Omega)\le\cdots\nearrow\infty$. The eigenfunctions associated to λ_k are denoted by φ_k. The quantities (λ_k,φ_k) depend on the geometry and size of Ω. They are known only for some particular domains including rectangles and circular sectors. This situation justifies the huge amount of mathematical papers dedicated to the derivation of lower and upper bounds for the eigenvalues λ_k in terms of geometric quantities as, for instance, the area $|\Omega|$ of Ω, the maximal conformal radius $\mathring{r}(\Omega)$, etc. Some books and review papers on this matter are referenced in C. Bandle [1], B. Dittmar [2], A. Henrot [3], L.E. Payne [4] or G. Pólya and G. Szegő [5].

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First we are going to review some isoperimetric inequalities closely related to our results. The first eigenvalue λ_1 satisfies the following pair of isoperimetric inequalities:

$$\frac{\pi j_{0,1}^2}{|\Omega|} \le \lambda_1(\Omega) \le \frac{j_{0,1}^2}{\left(\overset{\circ}{r}(\Omega)\right)^2}, \tag{1.2}$$

where $j_{0,1} \cong 2.4048$ is the first zero of the Bessel function $J_0(\mathbf{x})$ and $\overset{\circ}{r}(\Omega)$ is the maximal conformal radius of Ω. The first inequality (1.2) is the famous Rayleigh–Faber–Krahn inequality (see G. Faber [6] and E. Krahn [7]), while the second inequality was obtained by G. Pólya and G. Szegö [5] in 1951. Both inequalities are isoperimetric, with equality if and only if Ω is a disc. In 1956, L.E. Payne, G. Pólya and H.F. Weinberger [8] conjectured that the ratios $\frac{\lambda_2}{\lambda_1}$ and $\frac{\lambda_2+\lambda_3}{\lambda_1}$ attain their greatest value when Ω is a disc, i.e.

$$\frac{\lambda_2}{\lambda_1}(\Omega) \le \frac{j_{1,1}^2}{j_{0,1}^2} \cong 2.539, \tag{1.3}$$

$$\frac{\lambda_2+\lambda_3}{\lambda_1}(\Omega) \le 2\frac{j_{1,1}^2}{j_{0,1}^2} \cong 5.077, \tag{1.4}$$

where $j_{1,1} \cong 3.832$ is the first zero of the Bessel function $J_1(\mathbf{x})$. Inequality (1.3) was established by M.S. Ashbaugh and R.D. Benguria [9] in 1991, but (1.4) is still open. Further isoperimetric inequalities involving higher eigenvalues have been established for simply connected symmetric domains. $\Omega \subset \mathbb{R}^2$ is said symmetric of order $q \ge 2$ if Ω coincides with itself after a rotation of angle $\frac{2\pi}{q}$ about same point $\mathbf{O}$ called the center of symmetry of Ω. In 1965, J. Hersch [10] has established the following isoperimetric inequalities. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain symmetric of order $q = 2$, then we have

$$\left(\frac{1}{\lambda_2(\Omega)} + \frac{1}{\lambda_3(\Omega)}\right)\frac{1}{r_o^2(\Omega)} \ge \frac{2}{j_{1,1}^2} \cong 0.136221. \tag{1.5}$$

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain symmetric of order $q = 4$, then we have

$$\left(\frac{1}{\lambda_4(\Omega)} + \frac{1}{\lambda_5(\Omega)}\right)\frac{1}{r_o^2(\Omega)} \ge \frac{2}{j_{2,1}^2} \cong 0.07583, \tag{1.6}$$

where $j_{2,1} \cong 5.136$ is the first zero of the Bessel function $J_2(\mathbf{x})$. In (1.5), (1.6) we have equality if Ω is a disc and $r_o(\Omega)$ is the conformal radius of Ω at the center of symmetry $\mathbf{O} \in \Omega$. Note that $r_o(\Omega) = \overset{\circ}{r}$ for convex Ω as indicated in G. Pólya and G. Szegő [5], p. 22. J. Hersch has observed in [10] that the isoperimetric inequality

$$[\lambda_2(\Omega) + \lambda_3(\Omega)]\,\overset{\circ}{r}{}^2 \le 2j_{1,1}^2 \cong 29.3684, \tag{1.7}$$

would follow from the second inequality (1.2) together with (1.4) if this conjecture could be established. In Section 2 of this note, (1.7) will be established for symmetric domains Ω of order $q = 2$, i.e.

$$[\lambda_2(\Omega) + \lambda_3(\Omega)]\,r_o^2(\Omega) \le 2j_{1,1}^2 \cong 29.3684. \tag{1.8}$$

In Section 3, we establish the isoperimetric inequality

$$[\lambda_4(\Omega) + \lambda_5(\Omega)]\,r_o^2(\Omega) \le 2j_{2,1}^2 \cong 52.7569, \tag{1.9}$$

valid in the class of symmetric domains of order $q = 4$. Inequalities (1.8) and (1.9) are sharper than (1.5), (1.6) in view of the inequality

$$\left(\frac{1}{a} + \frac{1}{b}\right)(a+b) \ge 4, \tag{1.10}$$

valid for arbitrary positive a, b, with equality if and only if $a = b$. We note that isoperimetric inequalities analogous to (1.8), (1.9) for Neumann eigenvalues have been derived by C. Enache and G.A. Philippin in [11,12].