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Differential Harnack estimate for a semilinear parabolic equation on hyperbolic space

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ABSTRACT

In this paper, we obtain a differential Harnack estimate for a semilinear parabolic equation on hyperbolic space. As applications of this estimate, we prove a blow-up theorem for this equation and integrate along space–time to derive a classical Harnack inequality.

1. Introduction

In this paper, we study the following Cauchy problem in the n-dimensional hyperbolic space $\mathbb{H}^n$:

$$\begin{cases} f_t = \Delta_{\mathrm{H}} f + e^{\mu t} f^p, & \text{in } \mathbb{H}^n \times (0, +\infty), \\ f(x, 0) = f_0, & \text{in } \mathbb{H}^n, \end{cases} \tag{1.1}$$

where $p > 1$, $\mu > 0$.

In [1], the hyperbolic space $\mathbb{H}^n$ is equivalent to the unit ball $B_1 \subset \mathbb{R}^n$ endowed with the Poincaré metric

$$ds^2 = \frac{4}{(1 - |x|^2)^2} dx^2.$$

The geodesic distance between any $x \in \mathbb{H}^n$ and 0 is given by

$$d(x, 0) := \int_0^{|x|} \frac{2}{1 - s^2} ds = \ln\Big(\frac{1 + |x|}{1 - |x|}\Big).$$

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Set $\rho(x) = d(x, 0)$. The volume element of $\mathbb{H}^n$ is

$$d\mu = \frac{2^n}{(1 - |x|^2)^n} dx_1 \cdots dx_n = (\sinh \rho)^{n-1} d\rho d\theta,$$

$dx_1 \cdots dx_n = dx$ being the Lebesgue measure in $\mathbb{R}^n$ and (ρ, θ) being polar geodesic coordinates in $\mathbb{H}^n \setminus \{0\}$.

Then the Laplace–Beltrami operator is given by the following equalities

$$\Delta_{\mathrm{H}} = \frac{1}{4}(1 - |x|^2)^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{n-2}{2}(1 - |x|^2) \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

and

$$\Delta_{\mathrm{H}} = \frac{\partial^2}{\partial \rho^2} + (n-1) \coth \rho \frac{\partial}{\partial \rho} + \frac{1}{(\sinh \rho)^2} \Delta_\theta,$$

Δ_θ being the Laplace–Beltrami operator on the $n-1$-dimensional sphere of $\mathbb{R}^n$.

P. Li and S.-T. Yau in [2] were first pioneers to the study of differential Harnack inequalities which was brought to general parabolic geometric flows by R. Hamilton (see [3]). The importance of parabolic Harnack inequalities is well introduced in [4–7]. Using these inequalities one can derive ancient solutions, bounds on gradient Ricci solitons, Hölder continuity. A more sophisticated application of differential Harnack inequalities in geometry can be found in [8]. Harnack inequalities have been applied to the study of log-Sobolev constants (see [9–11]).

One of the main results is to suggest that the method developed in geometric flows can be used for blow-up of solutions for nonlinear parabolic equations on hyperbolic space. C. Bandle et al. [12] obtained that if $1 < p < 1 + \frac{\mu}{\lambda_0}$ ($\lambda_0 = \frac{(n-1)^2}{4}$ the infimum of the L^2-spectrum of $-\Delta_{\mathrm{H}}$), then every nontrivial solution of problem (1.1) blows up in finite time.

For the convenience, Δ and ∇ in the paper are the operators of the hyperbolic space. Without loss of generality, we may assume that the sectional curvature of $\mathbb{H}^n$ is -1. Let $f(x,t)$ be a positive smooth solution to (1.1) and $u := \log f$. The main object of our study is the following Harnack quantity

$$H \equiv \alpha \Delta u + \beta |\nabla u|^2 + c\mathrm{e}^{\mu t + (p-1)u} + \psi(t) + \phi(x), \tag{1.2}$$

where α, β, $c \in \mathbb{R}$, $\alpha > \beta$ and ψ, ϕ will be chosen suitably later.

We will derive our differential Harnack estimate.

Theorem 1.1. *Let $f(x,t)$ be a positive classical solution to (1.1), and $u(x,t) := \log f$. If α, β and c satisfy*

$$\alpha \geq 1, \ \alpha > \beta > 0, \quad \frac{\alpha(p-1) + 2\beta}{p} \geq c \geq \frac{(p-1)n\alpha^2}{4(\alpha - \beta)} > 0,$$

then we have

$$H \equiv \alpha \Delta u + \beta |\nabla u|^2 + c\mathrm{e}^{\mu t + (p-1)u} + \frac{n}{2(\alpha - \beta)(1 - \mathrm{e}^{-t})} + \frac{n(n-1)\alpha^2}{2\beta} \geq 0$$

for all t.

The paper is organized as follows. In Section 2 we prove Theorem 1.1 which describes differential Harnack estimate. There are applications of Theorem 1.1 in Section 3.

2. Harnack estimate

In this section, we shall first obtain our differential Harnack inequality, relying on the parabolic maximum principle.