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A class of nonstandard numerical methods for autonomous dynamical systems

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ABSTRACT

A nonstandard finite difference method for solving autonomous dynamical systems is constructed. The proposed numerical method is computationally efficient and easy to implement. It is designed so that it preserves positivity of solutions and the local behavior of the dynamical system near equilibria.

1. Introduction

Consider an n-dimensional dynamical system of the form

$$\frac{dx}{dt} = f(x); \qquad x(0) \in \mathbb{R}^n_+, \tag{1}$$

whose solutions are of the form $x = [x^1, x^2, \ldots, x^n]^T : \mathbb{R}_+ \to \mathbb{R}^n_+$ and f is a C^2 function that satisfies conditions which guarantee that $\mathbb{R}^n_+$ is positively invariant (e.g. see Appendix B of [1]). The set $\mathbb{R}^n_+$ is defined as

$$\mathbb{R}^n_+ := \{(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n : x^1 \geq 0, x^2 \geq 0, \ldots, x^n \geq 0\},$$

and it is assumed that the system (1) has a finite number of equilibria, each of which is hyperbolic.

There have been several successful attempts at designing numerical methods which preserve positivity of solutions and the local behavior, but only for special classes of systems of ordinary differential equations which exhibit certain structure (e.g. [2–9]). Here, a finite difference method is constructed in Section 2 which preserves the aforementioned qualitative properties for the system (1). The method is analyzed in Section 3 and a numerical example is presented in Section 4 to show the utility of the method. In Section 5, some concluding remarks are given.

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2. Construction of the numerical method

A finite difference method to approximate the system (1) can be written as

$$D_h(x_k) = F_h(f; x_k), \tag{2}$$

where $D_h(x_k) \approx \frac{dx}{dt}\big|_{t=t_k}$, $x_k \approx x(t_k)$, $F_h(f; x_k)$ approximates $f(x)$ in the system (1) and $t_k = t_0 + kh$, where $h > 0$.

The positive and elementary stable properties [10] of a numerical method are defined below:

Definition 1. The finite difference method (2) is called *positive*, if, for any value of the step size h, and $x_0 \in \mathbb{R}^n_+$ its solution remains positive, i.e., $x_k \in \mathbb{R}^n_+$ for all $k \in \mathbb{N}$.

Definition 2. The finite difference method (2) is called *elementary stable* if, for any value of the step size h, its fixed points $\bar{x}$ are the same as the equilibria of the differential system (1) and the local stability properties of each $\bar{x}$ are the same for both the differential system and the difference method.

For a numerical method to accurately approximate the system (1), it must always exhibit a nonnegative solution when the initial value is nonnegative. The need for elementary stability is derived from the fact that any approximation to the system (1) should exhibit the same local stability around equilibria, and should not introduce any spurious fixed points.

In order to design a numerical method which is both positive and elementary stable, the nonstandard finite difference (NSFD) method framework, first introduced by Mickens [11], is adopted. The following concise definition by Anguelov and Lubuma [10] is used:

Definition 3. The one-step finite-difference scheme (2) for solving System (1) is a NSFD method if at least one of the following conditions is satisfied:

- $D_h(x_k) = \frac{x_{k+1} - x_k}{\varphi(h)}$, where $\varphi(h) = h + \mathcal{O}(h^2)$ is a non-negative function;
- $F_h(f; x_k) = g(x_k, x_{k+1}, h)$, where $g(x_k, x_{k+1}, h)$ is a non-local approximation of the right-hand side of System (1).

It should be noted that a more robust and technical definition is given by Lubuma and Patidar in [12].

Using the above NSFD framework, the new numerical method which approximates the system (1) is constructed as follows:

$$\frac{x^i_{k+1} - x^i_k}{\phi(h)} = f^i(x_k)\omega^i_k, \tag{3}$$

where

$$\omega^i_k = \begin{cases} 1, & \text{if } f^i(x_k) \geq 0 \\ \dfrac{x^i_{k+1}}{x^i_k}, & \text{if } f^i(x_k) < 0 \end{cases} \tag{4}$$

for $i = 1, 2, \ldots, n$ and $\phi(h) = h + \mathcal{O}(h^2)$ is a denominator function which is to be determined in the next section.

The numerical treatment of the right-hand side function in the Method (3) is motivated by some of the discretization ideas of J. Benz et al. [13]. In [13], the authors propose a numerical method that works for both conservative and non-conservative systems by using a term similar to Eq. (4) which switches based on the conservativity-property of the system.