Applied Mathematics Letters 50 (2015) 98–104

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Rank constrained matrix best approximation problem

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ARTICLE INFO

Article history:
Received 28 March 2015
Received in revised form 8 June 2015
Accepted 9 June 2015
Available online 20 June 2015

Keywords:
Rank constrained matrix approximation
The Frobenius norm
Rank

ABSTRACT

In this paper, we study a rank constrained matrix approximation problem in the Frobenius norm:

$$\min_{r(X)=k} \|AXB - C\|_F^2$$

where k is a nonnegative integer. First, we derive the feasible interval of the parameter K for the existence of solutions to the problem. Second, on condition that such a solution exists, we give a general expression for the solution to the corresponding rank constrained matrix approximation problem. Last, we provide the feasible interval of the parameter K for the existence of the minimal norm of X.

1. Introduction

Throughout this paper, we adopt the following notation: the symbol $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices; $\mathbb{U}^{m\times m}$ denotes the set of all $m \times m$ unitary matrices; A^* and $r(A)$ stand for the conjugate transpose and rank of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_p denotes the p-by-p identity matrix; $r\begin{bmatrix} A, & B \end{bmatrix}$ stands for the rank of $\begin{bmatrix} A, & B \end{bmatrix}$; $A^\dagger$ stands for the *Moore–Penrose* inverse of $A \in \mathbb{C}^{m\times n}$, which is the unique matrix satisfying the equations (see [1])

$$(1)\ AXA = A, \qquad (2)\ XAX = X, \qquad (3)\ (AX)^* = AX, \qquad (4)\ (XA)^* = XA;$$

$X = A^-$ denotes a g-inverse of A which satisfies the equation $AXA = A$; E_A and F_A stand for the two orthogonal projections:

$$E_A = I_m - AA^\dagger \quad \text{and} \quad F_A = I_n - A^\dagger A.$$

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http://dx.doi.org/10.1016/j.aml.2015.06.009
0893-9659/

Furthermore, the symbol $\|A\|_F$ denotes the Frobenius norm of $A \in \mathbb{C}^{m\times n}$: $\|A\|_F^2 \overset{\text{def}}{=} \text{Trace}\,(A^*A)$. It is readily seen that the Frobenius norm is a unitarily invariant matrix norm.

Recently, findings in the research of rank matrix approximation problems have been so widely applied to signal process, control theory, numerical algebra, and so on [2–6, etc], that the problems have become a hot spot of investigation [7–11, etc]. Golub et al. [3] studied a rank constrained matrix approximation problem in all unitarily invariant norms: $\|[A, \quad X] - [A, \quad B]\|$ for X subject to $r([A, \quad X]) \le k$. Demmel [2] solved the following problem in the Frobenius norm and the 2-norm: $\min \left\| \begin{bmatrix} A & B \\ C & X \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\|$ for X subject to $r\left(\begin{bmatrix} A & B \\ C & X \end{bmatrix}\right) \le k$. Based on those results, Wei [12] presented perturbation analysis for the Eckart–Young–Mirsky theorem and the constrained total least squares problem. Wang [10] studied a generalisation of the above-mentioned problems. Sou & Rantzer [6], and Wei & Shen [11] studied minimum rank matrix approximation problems in the spectral norm and applied their findings to control theory. Sondermann [13], and Friedland & Torokhti [7] studied the following rank constrained matrix approximation:

$$\min_{r(X)\le k} \|A - BXC\|_F^2$$

and applied their findings to statistics and signal process, etc. Wang and Sun [9] studied the rank constrained matrix approximation: $\min \|A - BXC\|_F^2$ for X subject to $r(A - BXC) \le k$.

In [2,7,3,13], a common assumption is that the rank is less than or equal to a given non-negative integer. In fact, some problems are subject to the rank k, where k is equal to a given non-negative integer. For instance, matrix approximation is one of the most commonly used tasks in machine learning [4,5,14]. Given a partially observed matrix M, matrix completion constructs $\widehat{M}$ that approximates M at its unobserved entries. A general assumption is that M is of low-rank in constructing matrix approximations [8]. One of the most commonly used tasks is the incomplete SVD method, i.e., construct a low-rank approximation $\widehat{M} = UV^*$ such that

$$(U, V) = \arg\min_{U,V} \|\Pi_{\mathcal{A}}\left([UV^*] - M\right)\|_F^2,$$

or equivalently

$$\widehat{M} = \arg\min_{X} \|\Pi_{\mathcal{A}}\left(X - M\right)\|_F^2 \quad \text{s.t. } r(X) = k,$$

where the linear map $\Pi_{\mathcal{A}} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is given.

Notice that constraints $r(X) \le k$ and $r(X) = k$ are different. For instance, let $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ be a 2-by-2 matrix. When $r(X) \le 2$, x_1, x_2, x_3 and x_4 are all arbitrary; when $r(X) = 2$, x_1, x_2, x_3 and x_4 are arbitrary such that $x_1x_4 - x_2x_3 \ne 0$. Thus, the results of generalised matrix approximations with the rank less than or equal to K are supposed to be different from those with the rank equal to k.

We are, therefore, motivated to focus our research interest on rank constrained matrix approximation problem

$$\min_{r(X)=k} \|AXB - C\|_F^2, \tag{1.1}$$

where $A \in \mathbb{C}^{s\times m}$, $B \in \mathbb{C}^{n\times l}$ and $C \in \mathbb{C}^{s\times l}$.

2. Preliminaries

In this section, we present some preliminary results on the classical matrix approximation and rank inequalities in that the results will be used in the next section to study the rank constrained matrix approximation problem (1.1).