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An integrable nonlinear diffusion hierarchy with a two-dimensional arbitrary function

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ABSTRACT

By introducing the hereditary condition to a general first order differential strong symmetry operator, we obtain general 1+1 and 2+1 dimensional integrable nonlinear diffusion hierarchies with infinitely many symmetries and Lax pairs. For a special example the infinitely many nonlocal conservation laws and some explicit and implicit exact solutions are also given.

In physics and other scientific fields, there are many interesting problems that can be described by first order nonlinear partial differential equations (PDEs) which are special forms of

$$F(u, u_t, u_{x_1}, \ldots, u_{x_n}) = 0, \tag{1}$$

where F is an arbitrary function of the indicated variables and n is an arbitrary positive integer. For instance, the Hopf ($b = 0$) [1] and damped Hopf ($b \neq 0$) [2] equations (also named Burgers and damped Burgers equations),

$$u_t = auu_x + bu, \tag{2}$$

with arbitrary constants a and b are standard models to describe shock waves [3] with and without damping.

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The general Hopf equation, a model equation of gas dynamics,

$$u_t = f(u)u_x, \quad (f(u) \text{ being an arbitrary function of } u), \tag{3}$$

is also widely used in hydrodynamics, multiphase flows, wave theory, acoustics, chemical engineering and other applications [4,5].

The most general two dimensional form of (1),

$$F(u, u_x, u_y) = 0, \tag{4}$$

defines cylindrical surfaces whose elements are parallel to the xy plane [6].

One of the simple multiple dimensional significant models,

$$\sum_{i=0}^{n} a_i u_{x_i}^2 = c, \quad c, a_i, i = 0, 1, 2, \ldots, n, \text{ being arbitrary constants}, \tag{5}$$

is encountered in differential games [7].

In Ref. [8], it is proven that the general solution of (1) can be simply obtained by means of the symmetry theory. In other words, the first order autonomous PDE (1) is integrable for arbitrary F and n under the meaning that it possesses infinitely many symmetries. Now a natural important question is whether we can find integrable higher order PDEs from the general first order PDE system (1)?

To get some nontrivial results from the above question, we first restrict our attention to the simple case $n = 1$. So Eq. (1) can be rewritten as

$$u_t = u_x F, \tag{6}$$

where F is an arbitrary function of u and u_x. For the $(1 + 1)$-dimensional model (6), in Ref. [8], Lou and Yao constructed a strong symmetry operator which is expressed by

$$\Phi = \frac{u_x P}{u_{xx} + Q} \partial_x u_x^{-1}, \tag{7}$$

here $P \equiv P(u, u_x)$ and $Q \equiv Q(u, u_x)$ are the solutions of

$$\Delta = 0,\ \Delta \equiv u_x^2(F_u P_{u_x} - F_{u_x} P_u) + (2F_{u_x} + u_x F_{u_x u_x})QP - (u_x^2 F_{uu_x} + 2u_x F_u)P, \tag{8}$$

$$\Theta = 0,\ \Theta \equiv u_x^2(F_u Q_{u_x} - F_{u_x} Q_u) + (2F_{u_x} + u_x F_{u_x u_x})Q^2 - (2u_x^2 F_{uu_x} + 3u_x F_u)Q + u_x^3 F_{uu}. \tag{9}$$

In other words,

$$\sigma_n \equiv \Phi^n \sigma_0, \quad 0, \pm 1, \pm 2, \ldots, \tag{10}$$

are symmetries of (6) for any given seed symmetry σ_0, say, $\sigma_0 = u_x F$.

Now, the task to look for higher order integrable models is amounted to find certain possible constraints on Φ given in (7) such that it is a hereditary operator. As we know an operator Ψ is a hereditary (or Nijenhuis) operator if and only if

$$\Psi'[\Psi f]g - \Psi'[\Psi g]f - \Psi(\Psi'[f]g - \Psi'[g]f) = 0, \tag{11}$$

for arbitrary f and g. Here the Fréchet derivative of Ψ is defined by

$$\Psi'[h] \equiv \frac{d}{d\varepsilon} \Psi(u + \varepsilon h)\,|_{\varepsilon=0},$$

for any h.

It is shown in Ref. [9,10] if an operator Ψ is a hereditary operator, then

$$[\sigma_m, \sigma_n] \equiv \frac{d}{d\varepsilon} \left[\sigma_m(u + \varepsilon\sigma_n) - \sigma_n(u + \varepsilon\sigma_m)\right]\Big|_{\varepsilon=0} = 0, \qquad \sigma_m \equiv \Psi^m f, \tag{12}$$

for arbitrary $f = f(u, u_x, u_{xx}, \ldots)$.