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Lower bound for the lifespan of solutions for a class of fourth order wave equations

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ABSTRACT

This paper deals with blow-up solutions of a class of initial–boundary value problems for a fourth order semilinear wave equation. A lower bound for the lifespan of such solutions is derived.

1. Introduction

This paper deals with the following initial–boundary value problem for the semilinear Petrovsky equation

$$\begin{cases} u_{tt} + k_1 \Delta\Delta u + a u_t |u_t|^{m-2} = b u|u|^{p-2}, & x \in \Omega,\ t > 0, \\ u = 0, \qquad \dfrac{\partial u}{\partial n} = 0 \quad \text{or} \quad \Delta u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u_0(x),\ u_1(x)$ are suitable initial data and k_1, a, b, m, p are constants such that k_1 and b are positive, a nonnegative, and $m \geq 2, p > 2$. Ω is assumed to be bounded in R^n, $n = 2$ or 3, with smooth enough boundary $\partial\Omega$. Problem (1.1) is a model in various areas of mathematical physics as for instance in the theory of vibrating plates. The quantity $u_t|u_t|^{m-2}$ is a damping term in competition with the source term $u\ |u|^{p-2}$. Some further physical interpretation are given in [1] and [2].

The question of local existence of $u(x,t)$ has been investigated by Messaoudi in [3]. In the same paper Messaoudi showed that for $n < 4$, the solution blows up in finite time if $p > m \geq 2$ and if $E(0)$ is negative,

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where the energy $E(t)$ is defined as

$$E(t) := \frac{1}{2}\int_\Omega \left[u_t^2 + k_1(\Delta u)^2\right]\, dx - \frac{b}{a}\int_\Omega |u|^2 dx.$$

Moreover the solution is global if $m \geq p > 2$ for any given $u_0(x) \in H_0^2(\Omega)$ and $u_1(x) \in L^2(\Omega)$. The case $n \geq 5$ is also investigated. In a subsequent paper, Chen and Zhou [4] succeeded to show that the conditions for blow-up established by Messaoudi can be somewhat relaxed. In fact Chen and Zhou established that it is enough to assume $E(0) \leq 0$. Upper bounds for the lifespan t^* of blow-up solutions $u(x,t)$ are given in [4] and [3]. Further references relevant to problem (1.1) are [5], [6], [1]. However in practical situations it may be more interesting to dispose of some lower bound for the lifespan t^*. The goal of this note is to derive (in Section 2) such a lower bound for t^*. Some possible extension for the more general initial–boundary value problem

$$u_{tt} + k_1\Delta\Delta u - k_2\Delta u - k_3|\nabla u| + a u_t|u_t|^{m-2} = b u|u|^{p-2}, \quad x \in \Omega,\ t > 0, \tag{1.2}$$

with nonnegative constants k_2, k_3 under the initial and boundary conditions of (1.1) are indicated in Section 3. We note that the particular case of (1.2) with $k_1 = k_3 = 0$ has been investigated in [7].

2. A lower bound for t^*

In this section we want to derive a lower bound for the lifespan t^* of the blow-up solution of problem (1.1). To this end we introduce the auxiliary function

$$\Phi(t) := \int_\Omega \{u_t^2 + k_1(\Delta u)^2\} dx \tag{2.1}$$

and compute a value $T > 0$ such that $\Phi(t)$ remains bounded for $t \in [0, T]$. Clearly T is a lower bound for t^*. Differentiating (2.1) and making use of the second Green's formula, we obtain in view of (1.1)

$$\begin{aligned}\Phi'(t) &:= 2\int_\Omega \{u_t u_{tt} + k_1 \Delta u(\Delta u)_t\} dx = 2\int_\Omega u_t\{u_{tt} + k_1\Delta\Delta u\} dx \\ &= 2b\int_\Omega u_t u|u|^{p-2} dx - 2a\int_\Omega |u_t|^m dx \leq 2b\int_\Omega u_t u|u|^{p-2} dx.\end{aligned} \tag{2.2}$$

Making use of the Schwarz inequality leads to

$$\Phi'(t) \leq 2b\left(\int_\Omega u_t^2 dx \int_\Omega |u|^{2(p-1)} dx\right)^{\frac{1}{2}}. \tag{2.3}$$

Next, we make use of the following Lemma derived in [8].

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or $n = 3$, with smooth enough boundary $\partial\Omega$. Let $u(x)$ be a piecewise C^2-function, $x \in \Omega$ with $u(x) = 0$ on $\partial\Omega$. Let $p > 2$ be an arbitrary constant. Then the following inequality holds:*

$$\int_\Omega |u|^{2(p-1)} dx \leq \gamma\left(\int_\Omega (\Delta u)^2 dx\right)^{p-1}, \tag{2.4}$$

with

$$\gamma := \begin{cases} \left(\dfrac{p-1}{8}\right)^{2(p-1)} |\Omega|^p, & \text{for } n = 2, \\ \left(\dfrac{(p-1)^2}{12(p-2)}\right)^{2(p-1)} |\Omega|^{1+\frac{p-1}{3}}, & \text{for } n = 3, \end{cases} \tag{2.5}$$

where $|\Omega|$ is the measure of Ω, i.e. $|\Omega| = \int_\Omega dx$.