Applied Mathematics Letters 50 (2015) 146–152

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

On symmetry-preserving difference scheme to a generalized Benjamin equation and third-order Burgers equation ☆

Pan-Li Ma, Shou-Fu Tian *, Tian-Tian Zhang *

Department of Mathematics and Center of Nonlinear Equations, China University of Mining and Technology, Xuzhou 221116, People's Republic of China

ARTICLE INFO

Article history:
Received 21 May 2015
Received in revised form 23 June 2015
Accepted 23 June 2015
Available online 9 July 2015

Keywords:
Generalized Benjamin equation
Third-order Burgers equation
Lie point symmetries
Symmetry-preserving
Invariant difference models

ABSTRACT

In this paper, an exposition of a method is presented for discretizing a generalized Benjamin equation and third-order Burgers equation while preserving their Lie point symmetries. By using local conservation laws, the potential systems of original equation are obtained, which can be used to construct the invariant difference models and symmetry-preserving difference models of original equation, respectively. Furthermore, this method is very effective and can be applied to discrete high-order nonlinear evolution equations.

1. Introduction

It is well known that Lie symmetries play an important role in the analysis of nonlinear models. One of the main applications of the Lie group theory is to provide many powerful tools for solving ordinary and partial differential equations, specially nonlinear ones [1–3]. Based on Lie group theory, the standard way of solving ordinary and partial differential equations is to find the Lie point symmetry group G of the equation and then get invariant solutions. In addition, there are some effective methods to get group invariant solutions of differential equations, such as the classical Lie group approach and the non-classical Lie group approach [4–8].

Applications of the Lie group theory to difference equations are much more recent [9–25]. Lie group theory was originally invented as a systematic tool for obtaining exact analytical solutions of ordinary and partial differential equations (ODEs and PDEs). For differential equations the existence of a nontrivial symmetry group makes it possible to reduce the order of the equation. All numerical methods for solving ODEs replace the differential equation by a difference one, usually on a priori chosen lattice, either a regular one, or

☆ Project supported by the Fundamental Research Funds for the Central Universities of China under the Grant No. 2015QNA53.

* Corresponding authors.
E-mail addresses: sftian@cumt.edu.cn, shoufu2006@126.com (S.F. Tian), ttzhang@cumt.edu.cn (T.T. Zhang).

http://dx.doi.org/10.1016/j.aml.2015.06.017
0893-9659/

one adapted to some known or expected behavior of solutions. In this paper, we give a method to obtain the difference models completely preserving the symmetries of the original partial differential equation. The difference of this way to others is that we construct the symmetry-preserving difference models of the potential system instead of the original equation.

The paper is organized as follows: In Section 2, in order to make our presentation closed and self-contained, we briefly recall the overview of the discretization Procedure. In Section 3, the difference model of the generalized Benjamin equation is constructed which completely preserving the symmetries of the original partial differential equation. In Section 4, the same way is applied to the third-order Burgers equation and the difference model is constructed that the whole symmetries of original equation are preserved.

2. Preliminaries on discretization procedure

In order to make our presentation closed and self-contained, in this section, we briefly recall the required elementary notations [20–25] about the finite difference operators and Lie transformation groups.

In the space $\widetilde{Z}$ of formal series, consider the formal transformation group whose infinitesimal operator is the total derivative operator as follows

$$D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \cdots + u_{s+1} \frac{\partial}{\partial u_s} + \cdots . \tag{2.1}$$

For simplicity, in the following, we only consider the case of one independent variable x and one dependent variable u.

Let us fix an arbitrary parameter $h > 0$ and use the tangent field (2.1) of the Taylor group to form a pair of operators,

$$\underset{+h}{S} = e^{hD} \equiv \sum_{s=0}^{\infty} \frac{h^s}{s!} D^s, \qquad \underset{-h}{S} = e^{-hD} \equiv \sum_{s=0}^{\infty} \frac{(-h)^s}{s!} D^s, \tag{2.2}$$

the above operators be called the right and left discrete shift operators respectively, in which D is a derivation in $\widetilde{Z}$. By means of $\underset{+h}{S}$ and $\underset{-h}{S}$, we can form a pair of right and left discrete (finite-difference) differentiation operators as follows

$$\underset{+h}{D} = \frac{1}{h}(\underset{+h}{S} - 1) \equiv \sum_{s=1}^{\infty} \frac{h^{s-1}}{s!} D^s, \qquad \underset{-h}{D} = \frac{1}{h}(1 - \underset{-h}{S}) \equiv \sum_{s=1}^{\infty} \frac{(-h)^{s-1}}{s!} D^s. \tag{2.3}$$

Suppose

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \cdots , \tag{2.4}$$

is the generator of a one-parameter transformation group. For an evolution differential equation,

$$F(x, t, u, u_t, u_x, u_{xx}, u_{xxx}) = 0, \tag{2.5}$$

the group generated by (2.4) transformations a point $(x, t, u, u_t, u_x, u_{xx}, u_{xxx})$ to a new one $(x^*, t^*, u^*, u_t^*, u_x^*, u_{xx}^*, u_{xxx}^*)$ together with Eq. (2.5). When applying Lie point transformations to the difference equations, this situation be changed.

Proposition 1.1. *For an mesh $\underset{h}{\omega}$ to preserve uniform $(h_+ = h_-)$ under the action of the transformation group G_1, the following condition should be satisfied at each point $z \in \underset{h}{\widetilde{Z}}$:*

$$\underset{+h}{D}\,\underset{-h}{D}(\xi(z)) = 0. \tag{2.6}$$

The meshes satisfying criterion (2.6) are said to be invariantly uniform.