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Existence and uniqueness of the global solution to the Navier–Stokes equations

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ABSTRACT

A proof is given of the global existence and uniqueness of a weak solution to Navier–Stokes equations in unbounded exterior domains.

1. Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected C^2-smooth boundary S, and $D' := \mathbb{R}^3 \setminus D$ be the unbounded exterior domain.

Consider the Navier–Stokes equations:

$$u_t + (u, \nabla)u = -\nabla p + \nu \Delta u + f, \quad x \in D', \; t \geq 0, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u|_S = 0, \qquad u|_{t=0} = u_0(x). \tag{3}$$

Here f is a given vector-function, p is the pressure, $u = u(x,t)$ is the velocity vector-function, $\nu = const > 0$ is the viscosity coefficient, u_0 is the given initial velocity, $u_t := \partial_t u$, $(u, \nabla)u := u_a \partial_a u$, $\partial_a u := \frac{\partial u}{\partial x_a} := u_{;a}$, and $\nabla \cdot u_0 := u_{a;a} = 0$. Over the repeated indices a and b summation is understood, $1 \leq a, b \leq 3$. All functions are assumed real-valued.

We assume that $u \in W$,

$$W := \{u | L^2(0, T; H_0^1(D')) \cap L^\infty(0, T; L^2(D')) \cap u_t \in L^2(D' \times [0, T]); \nabla \cdot u = 0\},$$

where $T > 0$ is arbitrary.

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Let $(u, v) := \int_{D'} u_a v_a dx$ denote the inner product in $L^2(D')$, $\|u\| := (u, u)^{1/2}$. By u_{ja} the a-th component of the vector-function u_j is denoted, and $u_{ja;b}$ is the derivative $\frac{\partial u_{ja}}{\partial x_b}$. Eq. (2) can be written as $u_{a;a} = 0$ in these notations. We denote $\frac{\partial u^2}{\partial x_a} := (u^2)_{;a}$, $u^2 := u_b u_b$. By $c > 0$ various estimation constants are denoted.

Let us define a weak solution to problem (1)–(3) as an element of W which satisfies the identity:

$$(u_t, v) + (u_a u_{b;a}, v_b) + \nu(\nabla u, \nabla v) = (f, v), \quad \forall v \in W. \tag{4}$$

Here we took into account that $-(\Delta u, v) = (\nabla u, \nabla v)$ and $(\nabla p, v) = -(p, v_{a;a}) = 0$ if $v \in H_0^1(D')$ and $\nabla \cdot v = 0$. Eq. (4) is equivalent to the integrated equation:

$$\int_0^t [(u_s, v) + (u_a u_{b;a}, v_b) + \nu(\nabla u, \nabla v)]ds = \int_0^t (f, v)ds, \quad \forall v \in W. \tag{$*$}$$

Eq. (4) implies Eq. $(*)$, and differentiating Eq. $(*)$ with respect to t one gets Eq. (4) for almost all $t \geq 0$.

The aim of this paper is to prove the global existence and uniqueness of the weak solution to the Navier–Stokes boundary problem, that is, solution in W existing for all $t \geq 0$. Let us assume that

$$\sup_{t\geq 0} \int_0^t \|f\|ds \leq c, \quad (u_0, u_0) \leq c. \tag{A}$$

Theorem 1. *If assumptions (A) hold and $u_0 \in H_0^1(D)$ satisfies Eq. (2), then there exists for all $t > 0$ a solution $u \in W$ to (4) and this solution is unique in W provided that $\|\nabla u\|^4 \in L^1_{loc}(0, \infty)$.*

In Section 2 we prove Theorem 1. There is a large literature on Navier–Stokes equations, of which we mention only [1,2]. The global existence and uniqueness of the solution to Navier–Stokes boundary problems has not yet been proved without additional assumptions. Our additional assumption is $\|\nabla u\|^4 \in L^1_{loc}(0, \infty)$. The history of this problem see, for example, in [1]. In [2] the uniqueness of the global solution to Navier–Stokes equations is established under the assumption $\|u\|^8_{L^4(D')} \in L^1_{loc}(0, \infty)$.

2. Proof of Theorem 1

Proof of Theorem 1. The steps of the proof are: (a) derivation of a priori estimates; (b) proof of the existence of the solution in W; (c) proof of the uniqueness of the solution in W.

(a) *Derivation of a priori estimates*

Take $v = u$ in (4). Then

$$(u_a u_{b;a}, u_b) = -(u_a u_b, u_{b;a}) = -\frac{1}{2}(u_a, (u^2)_{;a}) = \frac{1}{2}(u_{a;a}, u^2) = 0,$$

where the equation $u_{a;a} = 0$ was used. Thus, Eq. (4) with $v = u$ implies

$$\frac{1}{2}\partial_t(u, u) + \nu(\nabla u, \nabla u) = (f, u) \leq \|f\|\|u\|. \tag{5}$$

We will use the known inequality $\|u\|\|f\| \leq \epsilon\|u\|^2 + \frac{1}{4\epsilon}\|f\|^2$ with a small $\epsilon > 0$, and denote by $c > 0$ *various* estimation constants.

One gets from (5) the following estimate:

$$(u(t), u(t)) + 2\nu \int_0^t (\nabla u, \nabla u)ds \leq (u_0, u_0) + 2\int_0^t \|f\|ds \sup_{s\in[0,t]} \|u(s)\| \leq c + c \sup_{s\in[0,t]} \|u(s)\|. \tag{6}$$

Recall that assumptions (A) hold. Denote $\sup_{s\in[0,t]} \|u(s)\| := b(t)$. Then inequality (6) implies

$$b^2(t) \leq c + cb(t), \quad c = const > 0. \tag{7}$$