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Three positive solutions for one-dimensional p-Laplacian problem with sign-changing weight

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ABSTRACT

We show that one-dimensional p-Laplacian with a sign-changing weight which is subject to Dirichlet boundary condition has three positive solutions suggesting suitable conditions on the weight function and nonlinearity. Proofs are mainly based on the directions of a bifurcation.

1. Introduction and main result

In this paper, we are concerned with the existence of three positive solutions for the following problem

$$\begin{cases} (\varphi_p(u'(x)))' + \lambda h(x) f(u(x)) = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (P_\lambda)$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\lambda > 0$ is a parameter, $f \in C[0,\infty)$, $f(0) = 0$, $f(s) > 0$ for all $s > 0$ and $h \in C[0,1]$ will be specified later.

The existence of three positive solutions for (P_λ) was mainly studied by three positive solutions theorem by Amann [1] which is established on the method of sub-supersolution. For example, for one-dimensional p-Laplacian with a weight function, Lee–Kim–Lee [2] showed the existence of three positive solutions by a variant of Amann's theorem. It is worth noting that even they established the basic three positive solutions theorem for a sign-changing weight case, they could only apply it for a positive weight case. This means that

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it is hard to construct sub-supersolutions in the case of a sign-changing weight. Variational approach [3,4] can also be applied to get three solutions, however, this method does not guarantee positivity or nontriviality of all solutions at most cases. Recently, there is a result about Hénon type equation by the second author [5];

$$\begin{cases} u''(x) + |x|^l u^p(x) = 0, & x \in (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{H}$$

where $l \geq 0$ and $p > 1$. It is shown the existence of one positive even solution and two positive non-even solutions when $l(p-1) \geq 4$ for (H), by using Morse index argument and comparing its linearized problem. However, it seems hard to follow this argument directly for p-Laplacian since p-Laplacian's linearized problem contains a degeneracy and we cannot apply comparison type theorems.

Motivated on these studies, we shall allow a weight to be sign-changing and use a different method, so-called a bifurcation. Throughout the paper, we assume that

(H1) there exist $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$, $h(x) > 0$ on (x_1, x_2) and $h(x) \leq 0$ on $[0,1] \setminus [x_1, x_2]$.

Condition (H1) allows the cases: $h(x) > 0$ on $[0,a)$ and $h(x) < 0$ on $(a,1]$ for some $a \in (0,1)$ or $h(x) > 0$ on (a,b) and $h(x) < 0$ on $[0,a) \cup (b,1]$ for some a, $b \in (0,1)$. Let μ_1 be the first eigenvalue [6,7] of the following problem:

$$\begin{cases} (\varphi_p(u'(x)))' + \mu h(x) \varphi_p(u(x)) = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{E_μ}$$

Then the first eigenvalue μ_1 is the minimum of the Rayleigh quotient, that is,

$$\mu_1 = \inf \left\{ \frac{\int_0^1 |\phi'(x)|^p dx}{\int_0^1 h(x)|\phi(x)|^p dx} : \phi \in W_0^{1,p}(0,1), \int_0^1 h(x)|\phi(x)|^p dx > 0 \right\}.$$

Furthermore, we assume that

(F1) there exist $\alpha > 0$, $f_0 > 0$ and $f_1 > 0$ such that $\lim_{s \to 0^+} \frac{f(s) - f_0 s^{p-1}}{s^{p-1+\alpha}} = -f_1$;
(F2) $f_\infty := \lim_{s \to \infty} f(s)/s^{p-1} = 0$;
(F3) there exists $s_0 > 0$ such that

$$\min_{s \in [s_0, 2s_0]} \frac{f(s)}{s^{p-1}} \geq \frac{f_0(p-1)}{\mu_1 h_0} \left(\frac{\pi_p}{x_2 - x_1} \right)^p,$$

where

$$\pi_p := \frac{2\pi}{p \sin(\pi/p)}, \qquad h_0 = \min_{x \in \left[\frac{3x_1 + x_2}{4}, \frac{x_1 + 3x_2}{4}\right]} h(x).$$

For π_p, we also refer the reader to [8,9] and [10]. It is easy to find that if (F1) holds, then

$$\lim_{s \to 0^+} \frac{f(s)}{s^{p-1}} = f_0. \tag{1.1}$$

Moreover, if (1.1) and (F2) hold, then there exists $f^* > 0$ such that

$$f(s) \leq f^* s^{p-1}, \quad s \geq 0. \tag{1.2}$$