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Large order Reynolds expansions for the Navier–Stokes equations

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ABSTRACT

We consider the incompressible homogeneous Navier–Stokes (NS) equations on a torus (typically, in dimension 3); we improve previous results of Morosi and Pizzocchero (2014) on the approximation of the solution via an expansion in powers of the Reynolds number. More precisely, we propose this approximation technique in the C^∞ setting of Morosi and Pizzocchero (2015) and present new applications, based on a Python program for the symbolic computation of the expansion. The *a posteriori* analysis of the approximants constructed in this way indicates, amongst else, global existence of the exact NS solution when the Reynolds number is below an explicitly computable critical value, depending on the initial datum; some examples are given.

1. Introduction

In this letter we consider the Cauchy problem for the incompressible homogeneous Navier–Stokes (NS) equations on a d-dimensional torus (typically, with $d = 3$), in the C^∞ formulation described, e.g., in [1]. In the cited work and in the previous paper [2] (inspired by [3,4]) it was shown how to obtain quantitative estimates on the exact solution u of the NS Cauchy problem via the *a posteriori* analysis of an approximate solution; such estimates concern the interval of existence of u and its distance from the approximate solution, evaluated in terms of Sobolev norms. In certain cases, this approach ensures global existence of the exact solution. Here we are interested in approximate NS solutions of the form $u^N(t) = \sum_{j=0}^{N} R^j u_j(t)$, where R is the "mathematical" Reynolds number (the reciprocal of the kinematic viscosity) and the coefficients $u_j(t)$ are determined stipulating that the NS equations be satisfied up to an error $O(R^{N+1})$. This subject was already treated in [5]; the unique application considered therein was a Reynolds expansion of order $N = 5$ in dimension $d = 3$ with the Behr–Nečas–Wu vortex as initial datum, computed symbolically via Mathematica.

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The present letter and its extended arXiv version [6] contain some enrichments of the above results. First of all we point out that the approach of [5], devised to work in a Sobolev space of finite order, can in fact be adapted to the C^∞ framework of [1]; this ultimately allows to estimate the distance between the exact solution u and the approximant u^N via Sobolev norms of arbitrarily large order (see item (ii) of the forthcoming Proposition 3.2). As a second enrichment, we present new applications based on a Python program devised for this purpose. More precisely, we carry from the order $N = 5$ to $N = 20$ the Reynolds expansion for the BNW vortex and we propose the expansions for the Taylor–Green and Kida–Murakami vortices of orders $N = 20$ and $N = 12$, respectively. All these expansions are computed symbolically by our Python program. The a posteriori analysis of the above expansions indicates, amongst else, that the solution of the NS equations for anyone of the three vortices is global if R is below an explicitly given critical value.

Admittedly, our critical Reynolds numbers are below the ones characterizing the turbulent regime. However these bounds are based on a rigorous theoretical analysis, and they are fully quantitative; their small size reflects the generally poor state of the art in proving existence of global strong NS solutions. In the extended arXiv version [6] we make comparison with a number of previous works containing results of global existence for the $d = 3$ NS equations. Some of these works refer to initial data with fast oscillation in one direction [7–9], a class not containing the vortices analyzed here. The other papers [10,11,2,5,1,4,12] mentioned in the arXiv version have an intersection with the initial data discussed here, but the conditions for global existence presented therein, if rephrased in terms of Reynolds numbers, give critical values even smaller than the present ones (see page 6 of [6], items (i)–(v)); to save space here we only make a comparison with a result of global existence from [2,1], see the final lines of Section 4.

2. Some preliminaries

NS equations; Reynolds number. The NS equations for an incompressible homogeneous fluid with no external forces, periodic boundary conditions and initial datum u_* can be written as

$$\frac{\partial u}{\partial t} = \Delta u + R\,\mathscr{P}(u,u), \qquad u(x,0) = u_*(x)\ ; \qquad \mathscr{P}(v,w) := -\mathfrak{L}(v \bullet \partial w). \tag{2.1}$$

Here: $R := 1/\nu$, with $\nu \in (0,+\infty)$ the kinematic viscosity; $t := \nu \mathtt{t}$ where $\mathtt{t}$ is the physical time; $u = u(x,t)$ is the divergence free velocity field; the space variables $x = (x_s)_{s=1,\dots,d}$ belong to the torus $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$; $\Delta := \sum_{s=1}^d \partial_{ss}$ is the Laplacian; $(v \bullet \partial w)_r := \sum_{s=1}^d v_s \partial_s w_r$ for $r = 1,\dots,d$ and for all sufficiently regular velocity fields v, w on $\mathbf{T}^d$; $\mathfrak{L}$ is the Leray projection onto the space of divergence free vector fields. In the sequel we refer to R as the *mathematical* Reynolds number; a *physical* Reynolds number proportional to R is introduced at the end of the section with appropriate motivations.

NS functional setting. Consider the space $\mathbb{D}' := D'(\mathbf{T}^d, \mathbf{R}^d)$ formed by the $\mathbf{R}^d$-valued distributions on $\mathbf{T}^d$; each $v \in \mathbb{D}'$ has a weakly convergent Fourier expansion $v = \sum_{k\in\mathbf{Z}^d} v_k e_k$ where $e_k(x) := e^{ik\bullet x}$ and the coefficients $v_k (= \overline{v_{-k}}) \in \mathbf{C}^d$. Let us also write $\mathbb{L}^2$ for the space $L^2(\mathbf{T}^d, \mathbf{R}^d)$. The functional setting proposed in [2] for Eq. (2.1) is based mainly on the Sobolev spaces of finite order

$$\begin{aligned}\mathbb{H}^n_{\Sigma 0} &:= \{v \in \mathbb{D}' \mid \operatorname{div} v = 0,\ \langle v\rangle = 0,\ \sqrt{-\Delta}^{\,n} v \in \mathbb{L}^2\} \\ &= \left\{ v \in \mathbb{D}' \mid k \bullet v_k = 0\ \forall k,\ v_0 = 0,\ \sum_{k\in\mathbf{Z}^d\setminus\{0\}} |k|^{2n} |v_k| < +\infty \right\},\end{aligned} \tag{2.2}$$

where $n \in \mathbf{R}$ is suitably chosen (the subscripts $\Sigma, 0$ recall the above vanishing conditions for the divergence and for the mean value $\langle v\rangle$). For any real n, $\mathbb{H}^n_{\Sigma 0}$ is a Hilbert space with the inner product and the norm $\langle v|w\rangle_n := \langle \sqrt{-\Delta}^{\,n} v | \sqrt{-\Delta}^{\,n} w\rangle_{L^2} = (2\pi)^d \sum_{k\in\mathbf{Z}^d\setminus\{0\}} |k|^{2n} \overline{v_k} \bullet w_k$, $\|v\|_n := \sqrt{\langle v|v\rangle_n}$. It is known that the bilinear map $\mathscr{P}$ in Eq. (2.1) sends $\mathbb{H}^n_{\Sigma 0} \times \mathbb{H}^{n+1}_{\Sigma 0}$ to $\mathbb{H}^n_{\Sigma 0}$ for all real $n > d/2$, and that the following holds for