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On the existence of positive solutions for the second-order boundary value problem

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ABSTRACT

This paper is concerned with the existence of positive solutions to a second order boundary value problem. By imposing growth conditions on f and using a generalization of the Leggett–Williams fixed point theorem, we prove the existence of at least three symmetric positive solutions.

1. Introduction

In the past 20 years, there has been attention focused on the existence of positive solutions to boundary value problems for ordinary differential equations; see [1–15]. It is well known that the Krasnosel'skii [16] fixed point theorems and the Leggett–Williams [17] multiple fixed-point theorem play an extremely important role.

In this paper, we discuss the existence of at least three positive solutions to the following boundary value problem:

$$u''(t) + f(u(t)) = 0, \quad t \in [0,1], \tag{1.1}$$

$$u'(0) = 0, \qquad u(1) = 0, \tag{1.2}$$

where $f : \mathbb{R} \to [0,\infty)$ is continuous. A solution $u \in C^{(2)}[0,1]$ of (1.1), (1.2) is both nonnegative and concave on [0,1]. We impose growth conditions on f which allows us to apply the generalization of the Leggett–Williams fixed point theorem in finding three symmetric positive solutions of (1.1), (1.2).

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2. Preliminaries

In this section, we give some background material concerning cone theory in a Banach space, and we then state the generalization of the Leggett–Williams fixed-point theorem.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is a cone if it satisfies the following two conditions:

(i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;
(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone P in a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous, and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P in a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ, β, θ be nonnegative continuous convex functionals on P, and α, ψ be nonnegative continuous concave functionals on P. Then for nonnegative real numbers h, a, b, d and c, we define the following convex sets:

$$\begin{aligned}
&P(\gamma, c) = \{u \in P : \gamma(u) < c, \},\\
&P(\gamma, \alpha, a, c) = \{u \in P : a \leq \alpha(u),\ \gamma(u) \leq c\},\\
&Q(\gamma, \beta, d, c) = \{u \in P : \beta(u) \leq d,\ \gamma(u) \leq c\},\\
&P(\gamma, \theta, \alpha, a, b, c) = \{u \in P : a \leq \alpha(u), \theta(u) \leq b, \gamma(u) \leq c\},\\
&Q(\gamma, \beta, \psi, h, d, c) = \{u \in P : h \leq \psi(u), \beta(u) \leq d,\ \gamma(u) \leq c\}.
\end{aligned}$$

We consider the two-point boundary value problem

$$-u'' = h(t), \quad t \in [0, 1], \tag{2.1}$$

$$u'(0) = 0, \qquad u(1) = 0. \tag{2.2}$$

Lemma 2.1. *Let $h \in L^1[0, 1]$. Then the two-point boundary value problem (2.1) and (2.2) has a unique solution*

$$u(t) = \int_0^1 G(t, s)h(s)ds$$

where Green's function $G(t, s)$ is

$$G(t, s) = \begin{cases} 1 - t, & 0 \leq s \leq t \leq 1,\\ 1 - s, & 0 \leq t \leq s \leq 1.\end{cases}$$

The following is a generalization of the Leggett–Williams fixed-point theorem which will play an important role in the proof of our main results.